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Finitized Conformal Spectra of the Ising Model on the Klein Bottle and Möbius Strip

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We study the conformal spectra of the critical square lattice Ising model on the Klein bottle and Möbius strip using Yang–Baxter techniques and the solution of functional equations. In particular, we obtain expressions for the *finitized* conformal partition functions in terms of *finitized* Virasoro characters. This demonstrates that Yang–Baxter techniques and functional equations can be used to study the conformal spectra of more general exactly solvable lattice models in these topologies. The results rely on certain properties of the eigenvalues which are confirmed numerically.

KEY WORDS: Conformal field theory; Ising model; Klein bottle; Möbius strip.

1. INTRODUCTION

There has been much recent progress on understanding general conformal boundary conditions for rational theories on the cylinder[(1)] and torus[(2)] and their relation[(3)] to integrable boundary conditions on the lattice. It is therefore of some interest to extend this understanding to other topologies such as the Klein bottle and Möbius strip. However, while much is known[(4–6)] about conformal partition functions in these topologies from the viewpoint of string theory, very little is known about integrable lattice boundary conditions on the Klein bottle and Möbius strip. Indeed, the lattice Yang–Baxter techniques which are well developed for the torus and cylinder have not yet been exploited in other topologies.

In this paper, we use Yang–Baxter techniques to study the conformal spectra of the critical square lattice Ising model on the Klein bottle and Möbius strip shown in Fig. 1. Although the Ising model has been studied

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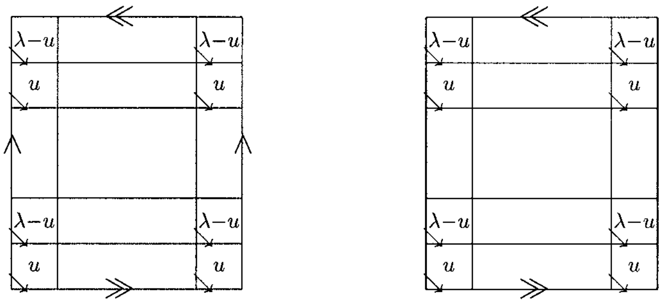


Fig. 1. Klein bottle (left) and Möbius strip (right). The boundaries are joined according to the orientations indicated by the arrows. The arrows at the bottom left corners on each face weight indicate its orientation.

previously in these topologies by free-fermion[7] and Pfaffian[8–10] techniques, we point out that these techniques do not generalize to other exactly solvable lattice models. In contrast, the methods based on Yang–Baxter techniques and the solution of functional equations developed in this paper will generalize to other exactly solvable lattice such as the *A*-*D*-*E* lattice models.[13]

The layout of the paper is as follows. In Section 2 we review the Ising model on the torus. We define the A_3 representation of the Ising model and its single row transfer matrix $\boldsymbol{T}(u)$. We also summarize the solution[14, 16] for the *finitized* conformal partition function using functional equations. In Section 3, we consider the Ising model on the Klein bottle. We define the double row transfer matrices $\boldsymbol{D}(u)=\boldsymbol{T}(u)\,\boldsymbol{T}(\lambda-u)$ and the flip operator $\boldsymbol{F}$ and discuss the simultaneous eigenvectors of $\boldsymbol{F}$ and $\boldsymbol{D}(u)$ and their relation to the eigenvectors of $\boldsymbol{T}(u)$. We calculate the conformal spectra of the Ising model on the Klein bottle in terms of *finitized* Virasoro characters by solving the relevant functional equation in the form of an inversion identity. In Section 4 we consider the Ising model on the Möbius strip. We define the double row transfer matrices $\boldsymbol{D}(u)$ with integrable boundary conditions corresponding to $+$ and free boundaries. The relevant inversion identities for the conformal spectra of the Ising model on the Möbius strip are the same as for the cylinder with $+$ or free boundary conditions on the left and right. For each eigenvalue, we identify the eigenvalue $F=\pm 1$ under the flip operator $\boldsymbol{F}$ according to the classifying pattern of 1- and 2-strings in the complex u-plane. This identification is not obtained analytically but is confirmed by extensive numerics determining the parity $F=\pm 1$ of the associated eigenvectors. Finally, we end with a brief discussion.

2. ISING MODEL ON THE TORUS

In this section we briefly review the critical Ising model with periodic boundary conditions on the torus and its finitized partition function.[12, 14, 16]

2.1. Ising Model

The Hamiltonian of the Ising model is given by

$$H(\{\sigma\}) = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i \tag{2.1}$$

where $\sigma_i = \pm 1$ is the spin at site i, $J_{ij} \geqslant 0$ the interaction strength between site i and j, and h the external field. The sum is over all pairs of sites $\langle i, j\rangle$.

In this article, we consider only the nearest-neighbour critical Ising model on square lattice with zero external field. The partition function is given by

$$Z(K, L) = \sum_{\{\sigma\}} \exp\left(K \sum_{\langle i,j\rangle_h} \sigma_i\sigma_j + L \sum_{\langle i,j\rangle_v} \sigma_i\sigma_j\right) \tag{2.2}$$

where K, L are the couplings in horizontal and vertical direction respectively. The model is critical on the line $\sinh(2K)\sinh(2L) = 1$. The partition function is a sum over all possible configurations of spins on the lattice with/without constraints due to the geometry of the lattice and the boundary conditions.

However, since the row-to-row transfer matrices in this representation do not commute, Baxter's commuting transfer matrices technique cannot be applied directly. Hence, we work in the A_3 representaion defined in the next sub-section for the following reasons:

1. The critical Ising model on a square lattice can be conveniently formulated as the critical A_3 model.
2. It can be easily generalised to other A-D-E models.

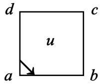
2.2. A_3 Representation and Its Transfer Matrices

The A_3 model is one of the Andrews–Baxter–Forrester models[17] A_L with $L = 3$ introduced in 1984. In 1987, Pasquier constructed the critical A-D-E models[13] which include the critical A_L models. In the A_3 model, the spins $a, b, c, d, \ldots$ assigned to the sites of the lattice take heights from the set $\{1, 2, 3\}$ and satisfy the adjacency condition that heights on adjacent sites must differ by ± 1.

Consider the Ising model on a square lattice of N columns and M rows where $N=2L$ and $M=2L'$ are both even.

The Boltzmann face weights of the critical A_3 model are

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\bigg| u \bigg) =$$



$$= \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{a,c} + \frac{\sin u}{\sin\lambda}\left(\frac{S_a S_c}{S_b S_d}\right)^{1/2}\delta_{b,d} \tag{2.3}$$

where $\lambda=\frac{\pi}{4}$ is the crossing parameter and $S_a=\sin(a\lambda)$ are the crossing factors. It is understood that the face weights vanish if the adjacency constraint $a-b=\pm 1$ is not satisfied along any edge. The spectral parameter u is usually taken in the range $0<u<\lambda$, so that the face weights are all positive but can be extended into the complex u-plane by analytic continuation.

The A_3 model corresponds to the two dimensional Ising model. In A_3 model, the square lattice has two sublattices which can be even or odd. The heights a are even on the even sublattice and odd on the odd sublattice. On the even sublattice, the heights are fixed to the value 2. On the odd sublattice we identify the state $a=1$ with the usual $+$ Ising state and $a=3$ with the usual $-$ Ising state. By the adjacency condition, if one of the sublattice is odd, then the other sublattice must be even and vice versa. The Ising model couplings act along the diagonals of the faces as depicted in Fig. 2. It can be shown that the spectral parameter u is related to the couplings by

$$\begin{aligned} e^{2K} &= W\left(\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\,\bigg|\, u\right)\bigg/ W\left(\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\,\bigg|\, u\right) = \frac{\sin(2\lambda-u)}{\sin u} \\ e^{2L} &= W\left(\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}\,\bigg|\, u\right)\bigg/ W\left(\begin{matrix} 3 & 2 \\ 2 & 1 \end{matrix}\,\bigg|\, u\right) = \frac{\sin(\lambda+u)}{\sin(\lambda-u)} \end{aligned} \tag{2.4}$$

It is obvious that the interactions are isotropic at $u=\frac{\lambda}{2}$. From this, we see that the A_3 model corresponds to the Ising model. Note that in the A_3 representation, it contains two mutually independent copies of Ising models. From now on, without stating explicitly, we use the term "Ising model" meaning "the Ising model in A_3 representation."

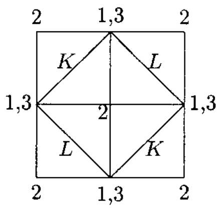


Fig. 2. Ising model couplings on A_3 lattice.

The face weights (2.3) have the following local properties. They are symmetric under reflection about the diagonals

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)=W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right)=W\left(\begin{matrix} d & a \\ c & b \end{matrix}\middle| u\right) \tag{2.5}$$

and satisfy the crossing symmetries

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)=\left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} W\left(\begin{matrix} c & b \\ d & a \end{matrix}\middle| \lambda-u\right)=\left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} W\left(\begin{matrix} a & d \\ b & c \end{matrix}\middle| \lambda-u\right) \tag{2.6}$$

By (2.5) and (2.6), we also have the following symmetries under horizontal and vertical reflections

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)=\left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| \lambda-u\right)=\left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} W\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| \lambda-u\right) \tag{2.7}$$

The single row transfer matrix $\boldsymbol{T}(u)$ between the rows of heights $\boldsymbol{a}=\{a_1,..., a_N\}$ and $\boldsymbol{b}=\{b_1,..., b_N\}$ has entries

$$\boldsymbol{T}(u)_{\boldsymbol{a},\boldsymbol{b}}=\prod_{j=1}^{N} W\left(\begin{matrix} b_j & b_{j+1} \\ a_j & a_{j+1} \end{matrix}\middle| u\right) \tag{2.8}$$

with the periodicity $a_{N+1}=a_1$, $b_{N+1}=b_1$.

Since the Boltzmann weights satisfy the Yang–Baxter equation and inversion relation, the transfer matrices $\boldsymbol{T}(u)$ form a commuting family[14, 21] with $\boldsymbol{T}(u)\,\boldsymbol{T}(v)=\boldsymbol{T}(v)\,\boldsymbol{T}(u)$. Using (2.7) and the periodic boundary condition along the row we find

$$\boldsymbol{T}(\lambda-u)_{\boldsymbol{a},\boldsymbol{b}}=\boldsymbol{T}(u)_{\boldsymbol{b},\boldsymbol{a}}=[\boldsymbol{T}^T(u)]_{\boldsymbol{a},\boldsymbol{b}} \tag{2.9}$$

So $\boldsymbol{T}(\lambda-u)=\boldsymbol{T}^T(u)$ and $\boldsymbol{T}(u)$ both belong to a commuting family of normal matrices which can be simultaneously diagonalized by a unitary matrix. The corresponding eigenvectors $\{z_n\}$ are in general complex and independent of u.

The adjacency matrix for the Ising model in the A_3 representation is

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.10}$$

i.e., $A_{ij}=1$ if heights i & j are admissible on adjacent sites and zero otherwise. It follows that the dimension of the transfer matrix is

$$\dim \boldsymbol{T}(u) = \mathrm{Tr}\, A^N = 2(2^{N/2}) = 2^{L+1} \tag{2.11}$$

Note that the dimension is twice larger than the one in usual representation as defined in Section 2.1 since we are working in A_3 representation which contains two copies of Ising models. In fact, we can order all the row configurations into two blocks. In the first block the first height in each row is even. In the second block, we cyclically translate the row configurations of the first block by one unit to the right so that the first height in each row is now odd. Using the fact that $\boldsymbol{T}(u)$ commutes with the shift operator $\boldsymbol{C}=\boldsymbol{T}(0)$ which is defined as

$$\boldsymbol{C}_{a,b} = \boldsymbol{T}(0)_{a,b} = \prod_{j=1}^{N} \delta_{a_j, b_{j+1}} \tag{2.12}$$

we obtain, in this basis, the block matrices

$$\boldsymbol{C} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ \tilde{\boldsymbol{C}} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{T}(u) = \begin{pmatrix} \mathbf{0} & \boldsymbol{V}(u) \\ \tilde{\boldsymbol{C}}\boldsymbol{V}(u) & \mathbf{0} \end{pmatrix} \tag{2.13}$$

where the orthogonal shift matrix $\tilde{\boldsymbol{C}}$ commutes with $\boldsymbol{V}(u)$ and each of the two blocks are of size 2^L. It follows that the eigenvalues of $\boldsymbol{T}(u)$ occur in pairs $\pm T(u)$. As an aside, $\boldsymbol{V}(u)$ and $\tilde{\boldsymbol{C}}\boldsymbol{V}(u)$ in (2.13), up to normalization, are equal to $\boldsymbol{V}(u)$ and $\boldsymbol{W}(u)$ as defined in Chapter 7 of ref. 14 respectively.

2.3. Finitized Ising Partition Function on the Torus

The critical properties of the Ising model are described by the unitary minimal conformal field theory with central charge $c=1/2$ and conformal weights $\Delta=0, 1/2$ and $1/16$. This conformal data is related to lattice quantities through finite-size corrections[18, 19] to the eigenvalues of the transfer matrices.

Consider the finite-size partition function Z_{NM} of the critical Ising model on a periodic lattice of N columns and M rows. The asymptotic behaviour of Z_{NM} in the limit of large N and M with the aspect ratio M/N fixed is given by

$$Z_{NM}^{\mathrm{Torus}}(u) = \mathrm{Tr}[\boldsymbol{T}(u)^M] \sim \exp[-NMf_{\mathrm{bulk}}(u)]\, Z^{\mathrm{Torus}}(q) \tag{2.14}$$

where $T(u)$ is the periodic row transfer matrix, $f_{\text{bulk}}(u)$ is the bulk free energy and $Z^{\text{Torus}}(q)$ is the universal conformal partition function with modular parameter

$$q = \exp\left(-2\pi i e^{-4iu} \frac{M}{N}\right) \tag{2.15}$$

Since the lattice is periodic, there is no boundary free energy.

The finite-size corrections of the eigenvalues $T(u)$ of the Ising model on a torus are given by refs. 18 and 19

$$\log T(u) = -N f_{\text{bulk}} + \frac{2\pi}{N}\left[\left(\frac{c}{12} - \Delta - \bar{\Delta}\right)\sin 4u - ike^{-4iu} + i\bar{k}e^{4iu}\right] + o\left(\frac{1}{N}\right) \tag{2.16}$$

where Δ, $\bar{\Delta}$ are conformal weights and k and $\bar{k}$ are arbitrary non-negative integers yielding towers of eigenvalues above each primary level $(\Delta, \bar{\Delta})$ with $k = \bar{k} = 0$. The modular invariant partition function is expressed in terms of Virasoro characters as

$$\begin{aligned} Z^{\text{Torus}}(q) &= \sum_{(\Delta,\bar{\Delta})} \mathcal{N}(\Delta, \bar{\Delta})\, q^{-c/12+\Delta+\bar{\Delta}} \sum_k q^k \sum_{\bar{k}} \bar{q}^{\bar{k}} \\ &= \chi_0(q)\,\chi_0(\bar{q}) + \chi_{1/16}(q)\,\chi_{1/16}(\bar{q}) + \chi_{1/2}(q)\,\chi_{1/2}(\bar{q}) \end{aligned} \tag{2.17}$$

where $\bar{q}$ is the complex conjugate of q and the operator content is given by

$$\mathcal{N}(\Delta, \bar{\Delta}) = \begin{cases} 1, & (\Delta, \bar{\Delta}) = (0,0), (1/16, 1/16), (1/2, 1/2) \\ 0, & \text{otherwise} \end{cases} \tag{2.18}$$

The $c = 1/2$ Virasoro characters are given variously by

$$\begin{aligned} q^{1/48}\chi_0(q) &= \frac{1}{2}\left\{\prod_{k=1}^{\infty}(1+q^{k-1/2}) + \prod_{k=1}^{\infty}(1-q^{k-1/2})\right\} \\ &= \sum_{\substack{m \geqslant 0 \\ m \text{ even}}} \frac{q^{\frac{1}{2}m^2}}{(q)_m} = \frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\{q^{j(12j+1)} - q^{(3j+1)(4j+1)}\} \end{aligned} \tag{2.19}$$

$$\begin{aligned} q^{-23/48}\chi_{1/2}(q) &= \frac{q^{-1/2}}{2}\left\{\prod_{k=1}^{\infty}(1+q^{k-1/2}) - \prod_{k=1}^{\infty}(1-q^{k-1/2})\right\} \\ &= \sum_{\substack{m \geqslant 0 \\ m \text{ odd}}} \frac{q^{\frac{1}{2}(m^2-1)}}{(q)_m} = \frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\{q^{j(12j+5)} - q^{(3j+2)(4j+1)}\} \end{aligned} \tag{2.20}$$

$$q^{-1/24}\chi_{1/16}(q) = \prod_{k=1}^{\infty}(1+q^k)$$
$$= \sum_{m\geqslant 0}\frac{q^{\frac{1}{2}m(m+1)}}{(q)_m} = \frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\{q^{j(12j-2)} - q^{(3j+1)(4j+2)}\} \qquad (2.21)$$

where $(q)_m = \prod_{k=1}^{m}(1-q^k)$ for $m>0$ and $(q)_0 = 1$. The three different forms of each of the characters constitute the $c = 1/2$ Rogers–Ramanujan identities.

For calculations, it is convenient to work with a *finitized* partition function normalized by the largest eigenvalue $T_0(u)$ for which $\Delta = \bar{\Delta} = k = \bar{k} = 0$

$$Z^{\text{Torus}}(L;q) = \sum_n \left(\frac{T_n}{T_0}\right)^M \to |q|^{c/12}\, Z^{\text{Torus}}(q), \qquad N, M\to\infty,\ M/N \text{fixed} \qquad (2.22)$$

where $n = 0, 1,\ldots$ labels the eigenvalues $T_n(u)$ of $\boldsymbol{T}(u)$ for $N = 2L$ columns. The contribution $|q|^{-c/12}$ from the largest eigenvalue $T_0(u)$ in the limit M and N large with M/N fixed is obtained in Appendix B by using the Euler–Maclaurin formula.

Following refs. 14 and 16, the eigenvalues $T(u)$ of the Ising model on a torus can be obtained by solving the functional equation

$$T(u)\,T(u+\tfrac{\pi}{4}) = \cos^{2L} 2u - R\sin^{2L} 2u \qquad (2.23)$$

where $L = N/2$, $T(u) = \overline{T(\frac{\pi}{4}-u)}$ and $\boldsymbol{T}(0)$ is the shift operator. Here $R = \pm 1$ is the eigenvalue of the spin or height reversal operator

$$\boldsymbol{R}_{a,b} = \prod_{j=1}^{N}\delta_{a_j, 4-b_j} \qquad (2.24)$$

which satisfies $\boldsymbol{R}^2 = \boldsymbol{I}$, $\boldsymbol{R}\boldsymbol{T}(u) = \boldsymbol{T}(u)\,\boldsymbol{R}$. For $R = 1$ the solution of the functional equations is[16]

$$T(u) = \epsilon\sqrt{2}\,(2e^{2iu+\pi i/4})^{-L}\prod_{k=1}^{L}\left(e^{4iu} + i\mu_k \tan\left(\frac{(2k-1)\,\pi}{4L}\right)\right) \qquad (2.25)$$

with $\epsilon^2 = \mu_k^2 = 1$ for all k and $\prod_{k=1}^{L}\mu_k = 1$. Similarly for $R = -1$, the solution is

$$T(u) = \epsilon\sqrt{L}\,(2e^{2iu+\pi i/4})^{1-L}\prod_{k=1}^{L-1}\left(e^{4iu} + i\mu_k \tan\left(\frac{k\pi}{2L}\right)\right) \qquad (2.26)$$

with $\epsilon^2 = \mu_k^2 = 1$ for all k. Up to an overall sign, the eigenvalues are thus determined by the set $\{\mu_k\}$. We note that the total number of eigenvalues is $2^L + 2^L = 2^{L+1}$ and that each eigenvalue appears with its negative.

It is often desirable to remove this trivial degeneracy of the eigenvalues $\pm T(u)$. Following ref. 16 it is convenient to choose

$$\epsilon = \prod_{k=\lfloor (L+3)/2 \rfloor}^{L} \mu_k, \quad R = +1; \qquad \epsilon = \prod_{k=\lfloor (L+2)/2 \rfloor}^{L-1} \mu_k, \quad R = -1 \tag{2.27}$$

This leads to the *finitized* Ising torus partition function[16]:

$$\begin{aligned} Z^{\text{Torus}}(L; q) = {} & X_0\left(\left\lfloor \frac{L}{2} \right\rfloor; \bar{q}\right) X_0\left(\left\lfloor \frac{L+1}{2} \right\rfloor; q\right) \\ & + |q|\, X_{1/2}\left(\left\lfloor \frac{L}{2} \right\rfloor; \bar{q}\right) X_{1/2}\left(\left\lfloor \frac{L+1}{2} \right\rfloor; q\right) \\ & + |q|^{1/8}\, X_{1/16}\left(\left\lfloor \frac{L-1}{2} \right\rfloor; \bar{q}\right) X_{1/16}\left(\left\lfloor \frac{L}{2} \right\rfloor; q\right) \end{aligned} \tag{2.28}$$

with finitized Virasoro characters[20]

$$\begin{aligned} X_0(L; q) &= \sum_{\{\mu\}_L^+} \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k,-1}} \\ &= \frac{1}{2} \left\{ \prod_{k=1}^{L} (1 + q^{k-1/2}) + \prod_{k=1}^{L} (1 - q^{k-1/2}) \right\} \\ &= \sum_{\substack{m=0 \\ m \text{ even}}}^{L} q^{m^2/2} \begin{bmatrix} L \\ m \end{bmatrix} \end{aligned} \tag{2.29}$$

$$\begin{aligned} X_{1/2}(L; q) &= q^{-1/2} \sum_{\{\mu\}_L} \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k,-1}} \\ &= \frac{1}{2} q^{-1/2} \left\{ \prod_{k=1}^{L} (1 + q^{k-1/2}) - \prod_{k=1}^{L} (1 - q^{k-1/2}) \right\} \\ &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{L} q^{(m^2-1)/2} \begin{bmatrix} L \\ m \end{bmatrix} \end{aligned} \tag{2.30}$$

$$\begin{aligned} X_{1/16}(L; q) &= \sum_{\{\mu\}_L} \prod_{k=1}^{L} \left(1 - \frac{1}{2} (1 - \mu_k)(1 - q^k) \right) \\ &= \prod_{k=1}^{L} (1 + q^k) = \sum_{m \geqslant 0}^{L} q^{m(m+1)/2} \begin{bmatrix} L \\ m \end{bmatrix} \end{aligned} \tag{2.31}$$

Here $\{\mu\}_L$ denotes the sequences $\mu_1,\ldots,\mu_L$ with $\mu_k=\pm 1$ and $\{\mu\}_L^{\pm}$ denotes the subset of sequences with $\prod_{k=1}^L \mu_k=\pm 1$. The Gaussian polynomials or q-binomials are

$$\begin{bmatrix} L \\ m \end{bmatrix} = \sum_{I_1=0}^{L-m} \sum_{I_2=0}^{I_1} \cdots \sum_{I_m=0}^{I_{m-1}} q^{I_1+\cdots+I_m} = \begin{cases} \dfrac{(q)_L}{(q)_m (q)_{L-m}}, & 0 \leqslant m \leqslant L \\ 0, & \text{otherwise} \end{cases} \tag{2.32}$$

with $(q)_m=\prod_{n=1}^m (1-q^n)$. The $R=1$ sector of the spectra is given by X_0 and $X_{1/2}$ and the $R=-1$ sector is given by $X_{1/16}$.

3. ISING MODEL ON THE KLEIN BOTTLE

3.1. Flip Operator and Partition Function

To form a Klein bottle we consider an $N\times M$ lattice of spins $a_{i,j}$ and impose periodic boundary conditions along the horizontal direction of the lattice by identifying column $N+1$ with column 1

$$a_{i,N+1}=a_{i,1}, \qquad i=1,2,\ldots,M \tag{3.1}$$

Along the vertical direction, we identify row $M+1$ with the left-right flip of row 1

$$a_{M+1,j}=a_{1,N+2-j}, \qquad j=1,2,\ldots,N \tag{3.2}$$

This flip reflects about the column $j=1$, or equivalently $j=\frac{N}{2}+1$, so that the spins at $j=1$ and $j=\frac{N}{2}+1=L+1$ are left invariant and the even and odd sublattices do not mix under the flip.

The flip operator $\boldsymbol{F}$ that implements the flip (3.2) has entries

$$\boldsymbol{F}_{a,b}=\prod_{j=1}^{N} \delta_{a_j, b_{N+2-j}} \tag{3.3}$$

This matrix has the same dimension as the transfer matrix $\boldsymbol{T}(u)$ and satisfies $\boldsymbol{F}^2=\boldsymbol{I}$, $\boldsymbol{F}^T=\boldsymbol{F}$ so that $\boldsymbol{F}$ has eigenvalues $F=\pm 1$. Clearly the flip operator $\boldsymbol{F}$ can be expressed, in a suitable basis, as a simple block matrix

$$\boldsymbol{F}=\begin{pmatrix} I_r & 0 & 0 \\ 0 & 0 & I_s \\ 0 & I_s & 0 \end{pmatrix} \tag{3.4}$$

where I_n is the $n\times n$ identity matrix, r is the number of row configurations which are invariant under the flip $\boldsymbol{F}$, $2s$ is the number of non-invariant row configurations which are related in pairs by the flip $\boldsymbol{F}$ and $r+2s=\dim \boldsymbol{F}=2(2^{N/2})=2^{L+1}$.

The number of left-right symmetric or invariant row configurations is simply the number of configurations of length $N/2$ and so

$$r=\sum_{a,\,b=1}^{3}\,[A^{N/2}]_{a,\,b},\qquad 2s=\mathrm{Tr}\,A^N-\sum_{a,\,b=1}^{3}\,[A^{N/2}]_{a,\,b} \tag{3.5}$$

Diagonalizing the matrix A and using spectral decomposition we find that

$$r=(\sqrt{2})^{\frac{N}{2}}\left(1+\frac{1}{\sqrt{2}}\right)^2+(-\sqrt{2})^{\frac{N}{2}}\left(1-\frac{1}{\sqrt{2}}\right)^2=\begin{cases}3(2^{L/2}), & L=\frac{N}{2}\text{ even}\\ 2^{(L+3)/2}, & L=\frac{N}{2}\text{ odd}\end{cases} \tag{3.6}$$

We see immediately that $\boldsymbol{F}$ has $r+s$ eigenvalues $+1$ and s eigenvalues -1 and so r is the net number of positive eigenvalues

$$r=\sum_n F_n \tag{3.7}$$

The flip operator $\boldsymbol{F}$ does not commute with the single row transfer matrix $\boldsymbol{T}(u)$ so it is not possible to simultaneously diagonalize $\boldsymbol{T}(u)$ and $\boldsymbol{F}$. Instead, we find

$$\begin{aligned}
[\boldsymbol{F}\boldsymbol{T}(u)]_{a,\,b}&=\prod_{j=1}^{N} W\left(\begin{matrix}b_j & b_{j+1}\\ a_{N+2-j} & a_{N+1-j}\end{matrix}\,\middle|\,u\right)\\
&=\prod_{j=1}^{N}\left(\frac{S_{a_{N+2-j}}S_{b_{j+1}}}{S_{a_{N+1-j}}S_{b_j}}\right)^{1/2} W\left(\begin{matrix}a_{N+2-j} & a_{N+1-j}\\ b_j & b_{j+1}\end{matrix}\,\middle|\,\lambda-u\right)\\
&=\prod_{j=1}^{N} W\left(\begin{matrix}a_{N+2-j} & a_{N+1-j}\\ b_j & b_{j+1}\end{matrix}\,\middle|\,\lambda-u\right)\\
&=[\boldsymbol{T}(\lambda-u)\,\boldsymbol{F}]_{a,\,b}=[\boldsymbol{T}^T(u)\,\boldsymbol{F}]_{a,\,b}
\end{aligned} \tag{3.8}$$

For this reason we introduce the double row transfer matrix $\boldsymbol{D}(u)$ defined by

$$\boldsymbol{D}(u)=\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)=\boldsymbol{T}^T(u)\,\boldsymbol{T}(u) \tag{3.9}$$

Clearly $\boldsymbol{D}(u)$ is real symmetric and positive definite for real u. Moreover, $\boldsymbol{F}$ and $\boldsymbol{D}(u)$ do commute so they can be simultaneously diagonalized

$$\begin{aligned} \boldsymbol{F}\boldsymbol{D}(u) &= \boldsymbol{F}\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u) = \boldsymbol{T}(u)\,\boldsymbol{F}\boldsymbol{T}(u) \\ &= \boldsymbol{T}(u)\,\boldsymbol{T}(\lambda-u)\,\boldsymbol{F} = \boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)\,\boldsymbol{F} = \boldsymbol{D}(u)\,\boldsymbol{F} \end{aligned} \tag{3.10}$$

The partition function of the Ising model on the Klein bottle is

$$Z_{MN}^{\text{Klein}}(u) = \text{Tr}[\boldsymbol{F}\boldsymbol{D}(u)^{M/2}] = \sum_n F_n D_n(u)^{M/2} \tag{3.11}$$

where n labels the eigenvalues with respect to the common set of eigenvectors $\boldsymbol{x}_n$ of $\boldsymbol{F}$ and $\boldsymbol{D}(u)$. Although the flip operator $\boldsymbol{F}$ reflects about the column $j=1$ we could have chosen a flip operator $\boldsymbol{F}_k$ that reflects about the column $j=k$. In this case

$$\boldsymbol{F}_k = \boldsymbol{C}^k\boldsymbol{F}\boldsymbol{C}^{-k} \tag{3.12}$$

It is then easy to check that the partition function (3.11) is translation invariant by using the cyclicity of the trace and the fact that $\boldsymbol{C}$ and $\boldsymbol{D}(u)$ commute

$$\begin{aligned} \text{Tr}[\boldsymbol{F}_k\boldsymbol{D}(u)^{M/2}] &= \text{Tr}[\boldsymbol{C}^k\boldsymbol{F}\boldsymbol{C}^{-k}\boldsymbol{D}(u)^{M/2}] = \text{Tr}[\boldsymbol{F}\boldsymbol{C}^{-k}\boldsymbol{D}(u)^{M/2}\,\boldsymbol{C}^k] \\ &= \text{Tr}[\boldsymbol{F}\boldsymbol{D}(u)^{M/2}] \end{aligned} \tag{3.13}$$

Hence we only need to work with the flip operator $\boldsymbol{F}=\boldsymbol{F}_1$ defined in (3.3).

We will show that the simultaneous eigenvectors of $\boldsymbol{F}$ and $\boldsymbol{D}(u)$ are

$$\{\boldsymbol{x}_n\} = \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r,\boldsymbol{z}_1^+,\ldots,\boldsymbol{z}_s^+,\boldsymbol{z}_1^-,\ldots,\boldsymbol{z}_s^-\} \tag{3.14}$$

with

$$\boldsymbol{F}\boldsymbol{x}_j = +\boldsymbol{x}_j,\quad j=1,2,\ldots,r;\qquad \boldsymbol{F}\boldsymbol{z}_k^\pm = \pm\boldsymbol{z}_k^\pm,\quad k=1,2,\ldots,s \tag{3.15}$$

so that $\boldsymbol{F}\boldsymbol{D}(u)^{M/2}\,\boldsymbol{z}_k^\pm = \pm\boldsymbol{D}(u)^{M/2}\,\boldsymbol{z}_k^\pm$. It follows that the eigenvalues corresponding to $\boldsymbol{z}_k^+$ and $\boldsymbol{z}_k^-$ cancel in (3.11). So only the eigenvalues of $\boldsymbol{D}(u)$ corresponding to the even eigenvectors $\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r\}$ contribute to the partition function. We will show that the eigenvectors $\{\boldsymbol{x}_j\}$ are precisely the real eigenvectors of $\boldsymbol{T}(u)$ whereas the eigenvectors $\{\boldsymbol{z}_k^\pm\}$ are simply related to the complex eigenvectors of $\boldsymbol{T}(u)$.

3.2. Simultaneous Eigenvectors of *D(u)* and *F*

In this section we study how the simultaneous eigenvectors of $\boldsymbol{D}(u)$ and $\boldsymbol{F}$ are related to the eigenvectors of $\boldsymbol{T}(u)$. Consider the transfer matrix $\boldsymbol{T}(u)$, with u in the closed interval $[0, \lambda] \subset \mathbb{R}$. This is a commuting family of real normal matrices that along with the flip operator $\boldsymbol{F}$ satisfies

$$\boldsymbol{T}(u)\, \boldsymbol{T}(v) = \boldsymbol{T}(v)\, \boldsymbol{T}(u), \qquad \boldsymbol{T}^T(u) = \boldsymbol{T}(\lambda-u), \qquad \boldsymbol{F}\boldsymbol{T}(\lambda-u) = \boldsymbol{T}(u)\, \boldsymbol{F}$$
$$\boldsymbol{D}(u) = \boldsymbol{T}^T(u)\, \boldsymbol{T}(u), \qquad \boldsymbol{F}\boldsymbol{D}(u) = \boldsymbol{D}(u)\, \boldsymbol{F} \qquad \boldsymbol{F}^2 = \boldsymbol{I} \tag{3.16}$$

The entries of $\boldsymbol{T}(u)$ are real analytic functions of u that can be analytically continued into the complex plane $u \in \mathbb{C}$. In general, the eigenvalues $T(u)$ and eigenvectors $\{z_n\}$ of $\boldsymbol{T}(u)$ are complex with the eigenvectors $z \in \{z_n\}$ being independent of u. Specifically, from (A.1) we have

$$\boldsymbol{T}(u)\, z = T(u)\, z, \qquad \boldsymbol{T}^T(u)\, z = \boldsymbol{T}(\lambda-u)\, z = T(\lambda-u)\, z = \overline{T(u)}\, z \tag{3.17}$$

The expressions for $T(u)$ can be obtained in the following way. Given an eigenvector $z = (z^1, z^2,...)$ with constant complex entries (since z is independent of u), we expand the product $\boldsymbol{T}(u)\, z$ along the row with the largest absolute value entry z^K in z. Since z is an eigenvector of $\boldsymbol{T}(u)$, the corresponding eigenvalue $T(u)$ is

$$T(u) = \frac{1}{z^K} \sum_j \, [\boldsymbol{T}(u)]_{K,j}\, z^j \tag{3.18}$$

Since $\boldsymbol{T}(u)$ is a finite matrix with entries being analytic functions, it follows that all the eigenvalues $T(u)$ which, are the finite sums of analytic functions, are analytic functions of u.

We say that two eigenvalues are degenerate if they agree on the whole interval $[0, \lambda]$. Clearly, by analytic continuation, two eigenvalues are degenerate if they agree on any open interval in $[0, \lambda]$. Consequently, non-degenerate eigenvalues can only cross at isolated points. The difference between these non-degenerate eigenvalues is an analytic function with isolated zeros which cannot accumulate in the closed interval $[0, \lambda]$. It follows that there is only a finite number of points where the crossings of the various eigenvalues can occur. Therefore at a "generic" point $u = u_0$ there are no accidental crossings, that is, any two eigenvalues are either degenerate on $[0, \lambda]$ or distinct at $u = u_0$. We conclude that the simultaneous eigenvectors of $\boldsymbol{T}(u)$ are uniquely determined (up to linear combinations for the degenerate eigenvalues) by the distinct eigenvectors at $u = u_0$.

So, in practice numerically, it is sufficient to fix u to a “generic” value u_0 and diagonalize $T(u_0)$ to find the simultaneous eigenvectors.

Let $\{z_n\}$ be a set of orthonomal complex eignvectors of $\boldsymbol{T}(u)$. We want to construct a set of simultaneous eigenvectors $\{\boldsymbol{x}_n\}$ of $\boldsymbol{D}(u)=\boldsymbol{T}^T(u)\,\boldsymbol{T}(u)$ and $\boldsymbol{F}$. Suppose, after suitable normalization, that the eigenvector $z=\boldsymbol{x}$ of $\boldsymbol{T}(u)$ is real. Then it follows that the corresponding eigenvalue $T(u)$ is real for all real u. We refer to these as the real eigenvalues of $\boldsymbol{T}(u)$.

$$\boldsymbol{T}(u)\,\boldsymbol{x}_j=\boldsymbol{T}^T(u)\,\boldsymbol{x}_j=T(u)\,\boldsymbol{x}_j,\qquad \boldsymbol{D}(u)\,\boldsymbol{x}_j=T(u)^2\,\boldsymbol{x}_j,\qquad j=1,2,\dots,r \tag{3.19}$$

In the remaining case we have to work with complex eigenvectors with both real and imaginary parts non-zero. In this case, the eigenvalues appear in complex conjugate pairs since z and $\bar{z}$ are linearly independent and $\boldsymbol{T}(u)\,z=T(u)\,z$ implies

$$\boldsymbol{T}(u)\,\bar{z}=\overline{\boldsymbol{T}(u)\,z}=\overline{T(u)\,z}=\overline{T(u)}\,\bar{z} \tag{3.20}$$

Hence

$$\boldsymbol{T}(u)\,z_k=T(u)\,z_k,\;\; \boldsymbol{T}^T(u)\,z_k=\overline{T(u)}\,z_k,\;\; \boldsymbol{D}(u)\,z_k=|T(u)|^2\,z_k,\;\; k=1,2,\dots,s \tag{3.21}$$

with a similar statement for $\bar{z}_k$.

Let $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_r,z_1,\dots,z_s\}$ be the set of mutually orthonormal real and complex eigenvectors corresponding to the real and complex eigenvalues $T(u_0)\in\mathbb{R}$, $T(u_0)\in\mathbb{C}$ with $\operatorname{Im}T(u_0)>0$. Then the $r+2s$ mutually orthonormal simultaneous eigenvectors of $\boldsymbol{D}(u)=\boldsymbol{T}^T(u)\,\boldsymbol{T}(u)$ and $\boldsymbol{F}$ are

$$\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_r,\boldsymbol{x}_1^+,\dots,\boldsymbol{x}_s^+,\boldsymbol{x}_1^-,\dots,\boldsymbol{x}_s^-\} \tag{3.22}$$

where

$$\boldsymbol{F}\boldsymbol{x}_j=F\boldsymbol{x}_j,\qquad \boldsymbol{x}_k^+=z_k^+,\qquad \boldsymbol{x}_k^-=-iz_k^-\qquad z_k^\pm=\frac{1}{\sqrt{2}}(\boldsymbol{I}\pm\boldsymbol{F})\,z_k \tag{3.23}$$

and the corresponding eigenvalues of $\{\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u),\boldsymbol{F}\}$ are $\{T(u)^2,F\}$, $\{|T(u)|^2,+1\}$ and $\{|T(u)|^2,-1\}$ respectively for the three groups. Although it is not apparent here we will see later that, since $\boldsymbol{F}z_k=\bar{z}_k$, the vectors $\boldsymbol{x}_j,\boldsymbol{x}_k^+,\boldsymbol{x}_k^-$ are all real with $\boldsymbol{x}_k^+=\sqrt{2}\operatorname{Re}(z_k)$, $\boldsymbol{x}_k^-=\sqrt{2}\operatorname{Im}(z_k)$.

We show that (3.22) is an orthonormal set of eigenvectors and hence a complete set of linearly independent eigenvectors of $\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)$ and $\boldsymbol{F}$. To establish this we note some properties of $\boldsymbol{T}(u)$ and $\boldsymbol{F}$ (see Appendix A):

- Two eigenvectors z_i and z_j of $\boldsymbol{T}(u)$ are orthogonal if their eigenvalues are distinct at $u=u_0$.
- If $\boldsymbol{T}(u)\,z=T(u)\,z$ then $\boldsymbol{F}z$ is also an eigenvector of $\boldsymbol{T}(u)$ with the corresponding eigenvalue $\overline{T(u)}$ since

$$\boldsymbol{T}(u)(\boldsymbol{F}\boldsymbol{x})=(\boldsymbol{T}(u)\,\boldsymbol{F})\,\boldsymbol{x}=(\boldsymbol{F}\boldsymbol{T}(\lambda-u))\boldsymbol{x}=\overline{T(u)}\,\boldsymbol{F}\boldsymbol{x} \tag{3.24}$$

Now we observe that the eigenvalues corresponding to the eigenvectors z_k and $\boldsymbol{F}z_\ell$ are distinct since they have imaginary parts $\operatorname{Im} T(u)>0$ and $\operatorname{Im}\overline{T(u)}<0$ respectively. So these two eigenvectors are orthogonal

$$z_k^\dagger \boldsymbol{F} z_\ell=0, \qquad k,\ell=1,2,\dots,s \tag{3.25}$$

The required orthonormality now follows by straigthforward calculation of inner products

$$(z_k^\pm)^\dagger\,\boldsymbol{x}_j=\frac{1}{\sqrt{2}}\,z_k^\dagger(\boldsymbol{I}\pm\boldsymbol{F})\,\boldsymbol{x}_j=\frac{1}{\sqrt{2}}\,(1\pm F)\,z_k^\dagger\boldsymbol{x}_j=0 \tag{3.26}$$

$$(z_k^\pm)^\dagger\,z_\ell^\pm=\frac{1}{2}\,z_k^\dagger(\boldsymbol{I}\pm\boldsymbol{F})^2\,z_\ell=z_k^\dagger(\boldsymbol{I}\pm\boldsymbol{F})\,z_\ell=z_k^\dagger z_\ell=\delta_{k,\ell} \tag{3.27}$$

$$(z_k^\pm)^\dagger\,z_\ell^\mp=\frac{1}{2}\,z_k^\dagger(\boldsymbol{I}\pm\boldsymbol{F})(\boldsymbol{I}\mp\boldsymbol{F})\,z_\ell=0 \tag{3.28}$$

Similarly, the required eigenvalues follow by straightforward calculation

$$\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)\,\boldsymbol{x}_j=\overline{T(u)}\,T(u)\,\boldsymbol{x}_j=T(u)^2\,\boldsymbol{x}_j, \qquad \boldsymbol{F}\boldsymbol{x}_j=F\boldsymbol{x}_j \tag{3.29}$$

$$\begin{aligned}\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)\,z_k^\pm&=\frac{1}{\sqrt{2}}\,\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)(\boldsymbol{I}\pm\boldsymbol{F})\,z_k\\&=\frac{1}{\sqrt{2}}\,\boldsymbol{T}(\lambda-u)(T(u)\,z_k\pm\overline{T(u)}\,\boldsymbol{F}z_k)\\&=\frac{1}{\sqrt{2}}\,(|T(u)|^2\,z_k\pm|T(u)|^2\,\boldsymbol{F}z_k)=|T(u)|^2\,z_k^\pm\end{aligned} \tag{3.30}$$

$$\boldsymbol{F}z_k^\pm=\frac{1}{\sqrt{2}}\,\boldsymbol{F}(\boldsymbol{I}\pm\boldsymbol{F})\,z_k=\pm\,\frac{1}{\sqrt{2}}\,(\boldsymbol{I}\pm\boldsymbol{F})\,z_k=\pm\,z_k^\pm \tag{3.31}$$

3.3. Eigenvectors of *T(u)*

To study the simultaneous eigenvectors of $\boldsymbol{T}(u)$ and $\boldsymbol{T}^T(u)$, we fix $u=u_0$ to a generic value to avoid accidental crossings of eigenvalues. We also set $\boldsymbol{T}=\boldsymbol{T}(u_0)$ and $T=T(u_0)$ so that the eigenvector equations for the common eigenvectors are

$$\boldsymbol{T}z=Tz,\qquad \boldsymbol{T}^Tz=\bar{T}z,\qquad \boldsymbol{D}z=\boldsymbol{T}^T\boldsymbol{T}z=|T|^2\,z \tag{3.32}$$

The transfer matrix $\boldsymbol{T}$ does not commute with the flip operator $\boldsymbol{F}$ but the double row transfer matrix $\boldsymbol{D}=\boldsymbol{T}^T\boldsymbol{T}$ does. In this section we obtain the structure of the complex eigenvectors z.

Let us order the $r+2s$ paths $\boldsymbol{a}$ into three blocks of size r, s and s respectively. In the first block we put the r paths that are invariant under the flip $\boldsymbol{F}$. The remaining $2s$ paths occur in pairs related by the flip and we place these in order in the second and third blocks. In this basis, the block structure of $\boldsymbol{F}$, $\boldsymbol{T}$ and $\boldsymbol{T}^T$ is

$$\boldsymbol{F}=\begin{pmatrix} I & 0 & 0\\ 0 & 0 & I\\ 0 & I & 0\end{pmatrix},\qquad \boldsymbol{T}=\begin{pmatrix} A & B & C\\ C^T & D & H\\ B^T & G & D^T\end{pmatrix},\qquad \boldsymbol{T}^T=\begin{pmatrix} A & C & B\\ B^T & D^T & G\\ C^T & H & D\end{pmatrix} \tag{3.33}$$

where $A^T=A$, $H^T=H$ and $G^T=G$ and we have used the fact that $\boldsymbol{F}\boldsymbol{T}=\boldsymbol{T}^T\boldsymbol{F}$.

The $r+s$ even and s odd real eigenvectors of $\boldsymbol{F}$ are of the form

$$\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\\ \boldsymbol{b}\end{pmatrix},\qquad \begin{pmatrix}0\\ \boldsymbol{c}\\ -\boldsymbol{c}\end{pmatrix} \tag{3.34}$$

It follows that the $r+s$ even and s odd common eigenvectors of $\boldsymbol{D}=\boldsymbol{T}^T\boldsymbol{T}$ and $\boldsymbol{F}$ are real linear combinations of these $r+s$ even and s odd eigenvectors respectively. Since the real symmetric matrices $\boldsymbol{D}=\boldsymbol{T}^T\boldsymbol{T}$ and $\boldsymbol{F}$ commute it follows that the real and imaginary parts of z in (3.32) are common real eigenvectors of $\boldsymbol{D}$ and $\boldsymbol{F}$ of the form (3.34). Consequently, there are three possibilities (i) z is even and $\boldsymbol{F}z=z$, (ii) z is odd and $\boldsymbol{F}z=-z$ or (iii) z is mixed so that after normalization $\boldsymbol{F}z=\bar{z}$ (In fact, it is not hard to show that $\boldsymbol{F}z=e^{i\theta}\bar{z}$ for any arbitrary real θ if z is mixed). But

$$\begin{aligned}\boldsymbol{T}z=Tz\Rightarrow \boldsymbol{T}^Tz=\bar{T}z \quad\text{and}\quad &\boldsymbol{T}^T(\boldsymbol{F}z)=T(\boldsymbol{F}z)\\ &\Rightarrow (T-\bar{T})\,z^\dagger(\boldsymbol{F}z)=0\;\Rightarrow\;\mathrm{Im}(T)\,z^\dagger(\boldsymbol{F}z)=0\end{aligned} \tag{3.35}$$

So if $T \notin \mathbb{R}$, we conclude that z and Fz are orthogonal. Hence $Fz \neq \pm z$ and consequently $Fz = \bar{z}$ and the complex eigenvectors of T occur in complex conjugate pairs

$$z = \begin{pmatrix} a \\ b+ic \\ b-ic \end{pmatrix}, \qquad Fz = \bar{z} = \begin{pmatrix} a \\ b-ic \\ b+ic \end{pmatrix} \tag{3.36}$$

We can write the eigenvector equations for T and T^T in terms of the symmetric and skew-symmetric combinations

$$(T+T^T)\, z = \begin{pmatrix} 2A & B+C & B+C \\ B^T+C^T & D+D^T & H+G \\ B^T+C^T & H+G & D+D^T \end{pmatrix} z = 2\,\mathrm{Re}(T)\, z \tag{3.37}$$

$$(T-T^T)\, z = \begin{pmatrix} 0 & B-C & C-B \\ C^T-B^T & D-D^T & H-G \\ B^T-C^T & G-H & D^T-D \end{pmatrix} z = 2i\,\mathrm{Im}(T)\, z \tag{3.38}$$

We now look for a complex eigenvector of the form (3.36) and take the real and imaginary parts in (3.37), (3.38)

$$\begin{pmatrix} 2A & B+C & B+C \\ B^T+C^T & D+D^T & H+G \\ B^T+C^T & H+G & D+D^T \end{pmatrix} \begin{pmatrix} a \\ b \\ b \end{pmatrix} = 2\,\mathrm{Re}(T) \begin{pmatrix} a \\ b \\ b \end{pmatrix} \tag{3.39}$$

$$\begin{pmatrix} 2A & B+C & B+C \\ B^T+C^T & D+D^T & H+G \\ B^T+C^T & H+G & D+D^T \end{pmatrix} \begin{pmatrix} 0 \\ c \\ -c \end{pmatrix} = 2\,\mathrm{Re}(T) \begin{pmatrix} 0 \\ c \\ -c \end{pmatrix} \tag{3.40}$$

$$\begin{pmatrix} 0 & B-C & C-B \\ C^T-B^T & D-D^T & H-G \\ B^T-C^T & G-H & D^T-D \end{pmatrix} \begin{pmatrix} a \\ b \\ b \end{pmatrix} = -2\,\mathrm{Im}(T) \begin{pmatrix} 0 \\ c \\ -c \end{pmatrix} \tag{3.41}$$

$$\begin{pmatrix} 0 & B-C & C-B \\ C^T-B^T & D-D^T & H-G \\ B^T-C^T & G-H & D^T-D \end{pmatrix} \begin{pmatrix} 0 \\ c \\ -c \end{pmatrix} = 2\,\mathrm{Im}(T) \begin{pmatrix} a \\ b \\ b \end{pmatrix} \tag{3.42}$$

The symmetric equations can be simplified and from the two skew-symmetric equations we obtain two eigenvector equations for real symmetric matrices

$$\begin{pmatrix} 2A & \sqrt{2}\,(B+C) \\ \sqrt{2}\,(B+C)^T & D+D^T+H+G \end{pmatrix}\begin{pmatrix} \boldsymbol{a} \\ \sqrt{2}\,\boldsymbol{b} \end{pmatrix} = 2\,\mathrm{Re}(T)\begin{pmatrix} \boldsymbol{a} \\ \sqrt{2}\,\boldsymbol{b} \end{pmatrix} \tag{3.43}$$

$$\begin{pmatrix} D+D^T & H+G \\ H+G & D+D^T \end{pmatrix}\begin{pmatrix} \boldsymbol{c} \\ -\boldsymbol{c} \end{pmatrix} = 2\,\mathrm{Re}(T)\begin{pmatrix} \boldsymbol{c} \\ -\boldsymbol{c} \end{pmatrix} \tag{3.44}$$

$$\begin{pmatrix} 0 & B-C & C-B \\ C^T-B^T & D-D^T & H-G \\ B^T-C^T & G-H & D^T-D \end{pmatrix}^2 \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{b} \end{pmatrix} = -4(\mathrm{Im}\,T)^2 \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{b} \end{pmatrix} \tag{3.45}$$

$$\begin{pmatrix} 0 & B-C & C-B \\ C^T-B^T & D-D^T & H-G \\ B^T-C^T & G-H & D^T-D \end{pmatrix}^2 \begin{pmatrix} 0 \\ \boldsymbol{c} \\ -\boldsymbol{c} \end{pmatrix} = -4(\mathrm{Im}\,T)^2 \begin{pmatrix} 0 \\ \boldsymbol{c} \\ -\boldsymbol{c} \end{pmatrix} \tag{3.46}$$

Equation (3.46) now reduces to a simple eigenvector equation for a real symmetric matrix

$$[2(B-C)^T\,(B-C)-(D-D^T)^2+(H-G)^2]\;\boldsymbol{c} = 4(\mathrm{Im}\,T)^2\;\boldsymbol{c} \tag{3.47}$$

From numerical observation, this equations yields s real eigenvectors with $\mathrm{Im}\,T \neq 0$ at any generic point u. This is verified in Section 3.5 by counting the number of real eigenvalues of $\boldsymbol{T}$ which matches the net number of positive eigenvalues of $\boldsymbol{F}$ in (3.6). Indeed, we will see that this generic situation leads to s complex conjugate pairs of eigenvectors of the form (3.36). A non-generic choice of u would lead to fewer than s complex conjugate eigenvalues and eigenvectors. From (3.44) we see that, in the generic situation, the s vectors $\boldsymbol{c}$ must also satisfy the simple real symmetric eigenvector equation

$$[D+D^T-H-G]\;\boldsymbol{c} = 2\,\mathrm{Re}(T)\;\boldsymbol{c} \tag{3.48}$$

For a given solution vector $\boldsymbol{c}$ we have thus determined the real and imaginary parts of the eigenvalues $T = \mathrm{Re}(T) \pm i\,\mathrm{Im}(T)$. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are determined by (3.42)

$$\boldsymbol{a} = (\mathrm{Im}\,T)^{-1}\,(B-C)\;\boldsymbol{c}, \quad \boldsymbol{b} = (2\,\mathrm{Im}\,T)^{-1}\,(D-D^T-H+G)\;\boldsymbol{c}, \quad \mathrm{Im}(T) \neq 0 \tag{3.49}$$

We have thus completely determined the s pairs of complex eigenvectors z, $\bar{z}$ of (3.36) corresponding to the complex eigenvalues $T = \mathrm{Re}(T) \pm i\,\mathrm{Im}(T)$ with $\mathrm{Im}(T) \neq 0$. The remaining equations such as (3.43) that we did not use in deriving the these eigenvectors must be automatically satisfied.

Each of the s complex conjugate pairs of eigenvectors gives rise to one even eigenvector $\boldsymbol{x}^+ = \sqrt{2}\,\mathrm{Re}(\boldsymbol{z})$ and one odd eigenvector $\boldsymbol{x}^- = \sqrt{2}\,\mathrm{Im}(\boldsymbol{z})$ of $\boldsymbol{F}$ and $\boldsymbol{D} = \boldsymbol{T}^T\boldsymbol{T}$. These completely exhaust the odd common eigenvectors of $\boldsymbol{F}$ and $\boldsymbol{D} = \boldsymbol{T}^T\boldsymbol{T}$. In the generic situation, it therefore remains to obtain the remaining r real even eigenvectors of $\boldsymbol{T}$

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{b} \end{pmatrix} \tag{3.50}$$

which necessarily satisfy

$$\boldsymbol{F}\boldsymbol{x} = \boldsymbol{x}, \qquad \boldsymbol{T}\boldsymbol{x} = \boldsymbol{T}^T\boldsymbol{x} = T\boldsymbol{x}, \qquad \boldsymbol{D}\boldsymbol{x} = \boldsymbol{T}^T\boldsymbol{T}\boldsymbol{x} = |T|^2\,\boldsymbol{x}, \qquad T \in \mathbb{R} \tag{3.51}$$

These eigenvectors are in fact given by the symmetric eigenvector equation (3.43) which yields $r+s$ even eigenvectors. The r solutions that we need are the solutions that remain after the s solutions for $(\boldsymbol{a}, \boldsymbol{b})$ obtained in the complex case are removed.

In a non-generic situation, such as $u = 0$ when $\boldsymbol{T}$ reduces to the shift operator, there are fewer than s complex conjugate pairs of eigenvectors of $\boldsymbol{T}$. In this case, each complex conjugate pair that is lost is replaced by one even and one odd real eigenvector. This is the only time that odd real eigenvectors of $\boldsymbol{T}$ occur. We therefore conclude that, in the generic situation, all eigenvectors $\boldsymbol{z}$ of $\boldsymbol{T}$ satisfy

$$\boldsymbol{F}\boldsymbol{z} = \bar{\boldsymbol{z}} \tag{3.52}$$

In this case the common real eigenvectors of $\boldsymbol{D} = \boldsymbol{T}^T\boldsymbol{T}$ and $\boldsymbol{F}$ are

$$\boldsymbol{x}^+ = \frac{1}{\sqrt{2}}(\boldsymbol{I}+\boldsymbol{F})\,\boldsymbol{z} = \sqrt{2}\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{b} \end{pmatrix} = \sqrt{2}\,\mathrm{Re}(\boldsymbol{z}) \tag{3.53}$$

$$\boldsymbol{x}^- = -\frac{i}{\sqrt{2}}(\boldsymbol{I}-\boldsymbol{F})\,\boldsymbol{z} = \sqrt{2}\begin{pmatrix} 0 \\ \boldsymbol{c} \\ -\boldsymbol{c} \end{pmatrix} = \sqrt{2}\,\mathrm{Im}(\boldsymbol{z}) \tag{3.54}$$

Clearly, these eigenvectors are of the form (3.34).

3.4. Conformal Finite-Size Corrections

Consider the finite-size partition function Z_{NM} of the critical Ising model on a Klein bottle formed from a lattice with N columns and M rows. The asymptotic behaviour of Z_{NM} in the limit of large N and M with the aspect ratio M/N fixed is given by

$$Z_{NM}^{\text{Klein}}(u) = \operatorname{Tr}[\boldsymbol{F}\boldsymbol{D}(u)^{M/2}] \sim \exp[-NMf_{\text{bulk}}(u) - Mf_{\text{bdy}}(u)]\, Z^{\text{Klein}}(q) \tag{3.55}$$

where $\boldsymbol{D}(u)$ is the double row transfer matrix, $f_{\text{bulk}}(u)$ is the bulk free energy, $f_{\text{bdy}}(u)$ is the boundary free energy and $Z^{\text{Klein}}(q)$ is the universal conformal partition function. Since we are working with a double row such that $D(u) = T(u)\, T(\lambda - u)$ and that $q_{\text{Torus}}(\lambda - u) = \overline{q_{\text{Torus}}(u)}$, the product $q_{\text{Torus}}(u)\, \overline{q_{\text{Torus}}(u)}$ is real positive and thus we define the modular parameter to be real and given by

$$q = \exp\left(-2\pi \frac{M}{N} \sin 4u\right) = |q_{\text{Torus}}| \tag{3.56}$$

where q_{Torus} is given by (2.15). Because of the presence of the flip, the left and right movers mix. Consequently, as is the case on the cylinder, there is only one copy of the Virasoro algebra. The leading corrections to each transfer matrix eigenvalue $D(u)$ are obtained from the torus case by identifying Δ with $\bar{\Delta}$ and k with $\bar{k}$ in (2.16)

$$\frac{1}{2}\log D(u) = -Nf_{\text{bulk}}(u) - f_{\text{bdy}}(u) + \frac{4\pi}{N}\left(\frac{c}{24} - \Delta - k\right)\sin(4u) + o\left(\frac{1}{N}\right) \tag{3.57}$$

with $\Delta = 0, 1/2, 1/16$ and k a non-negative integer. Cardy[23] argued that as the model is conformal invariant, all spectra are organized into Virasoro characters, and thus the eigenvalues naturally divide into three towers with

$$Z^{\text{Klein}}(q) = \sum_{\Delta} \mathcal{N}(\Delta)\, (q^2)^{-c/24+\Delta} \sum_{k} q^{2k} = \sum_{\Delta} \mathcal{N}(\Delta)\, \chi_{\Delta}(q^2) \tag{3.58}$$

where $\mathcal{N}(\Delta) \in \{0, 1, 2, \ldots\}$ is the operator content on the Klein bottle. Equation (3.58) with $\mathcal{N}(\Delta) = 1$ is confirmed numerically in the next section.

3.5. Partition Function of the Ising Model on the Klein Bottle

The *finitized* conformal partition function on the Klein bottle is

$$Z^{\text{Klein}}(L; q) = \sum_n F_n \left(\frac{D_n}{D_0}\right)^{M/2} \tag{3.59}$$

where $n = 0, 1, \ldots$ labels the eigenvalues D_n of $\boldsymbol{D}(u)$ and F_n of $\boldsymbol{F}$ for $N = 2L$ columns and D_0 is the largest eigenvalue. We will obtain this partition function from the eigenvalues $T(u)$ of the periodic row transfer matrix as given by (2.25), (2.26). To do this we need to separate out the r real eigenvalues from the $2s$ eigenvalues which appear in complex conjugate pairs.

It can be shown[14] that the eigenvalues $T(u)$ possess the periodicity

$$T\left(u + \frac{\pi}{2}\right) = (-1)^L T(u) \tag{3.60}$$

and that within the period strip $-\frac{\pi}{8} \leqslant \text{Re}(u) < \frac{7\pi}{8}$ in the complex u-plane the eigenvalues have exactly $2P$ zeros where

$$P = \begin{cases} L, & R = +1 \\ L-1, & R = -1 \end{cases} \tag{3.61}$$

For the ground state, given by $\mu_k = 1$ for all k, the zeros occur within the period strip as 2-strings on the lines $\text{Re}(u) = -\frac{\pi}{8}, \frac{3\pi}{8}$. For excitations, zeros also occur on the lines $\text{Re}(u) = \frac{\pi}{8}, \frac{5\pi}{8}$. The eigenvalues are in fact completely determined by the patterns of zeros in the complex u-plane. Indeed we will show that $T(u)$ can be uniquely expressed in the form

$$T(u) = R^{\pm} \prod_{k=1}^{P} \sin 2(u - u_k), \qquad R^{\pm}, u_1, \ldots, u_P \in \mathbb{C} \tag{3.62}$$

and obtain explicit expressions for the constant $R^{\pm}$ and zeros u_k.

Let us define the monotonically increasing sequences

$$t_k = \begin{cases} \tan\left(\dfrac{(2k-1)\,\pi}{4L}\right), & R = +1, \quad k = 1, \ldots, L \\[2ex] \tan\left(\dfrac{k\pi}{2L}\right), & R = -1, \quad k = 1, \ldots, L-1 \end{cases} \tag{3.63}$$

so that

$$t_k = t_{P+1-k}^{-1}, \qquad k = 1,\ldots, P \tag{3.64}$$

$$t_{\lceil \frac{P}{2} \rceil} = 1, \quad P \text{ odd}; \qquad \prod_{k=1}^{P} t_k = 1 \tag{3.65}$$

For $R = +1$, we can now re-express (2.25) in the form

$$\begin{aligned}
T(u) &= \epsilon \sqrt{2}\,(2e^{2iu+\frac{\pi i}{4}})^{-L} \prod_{k=1}^{L} \left(e^{4iu} + i\mu_k \tan\left(\frac{(2k-1)\,\pi}{4L}\right)\right) \\
&= \epsilon \sqrt{2} \prod_{k=1}^{L} \frac{1}{2i} [e^{2iu+\frac{\pi i}{4}} + t_k e^{\frac{\pi i}{2}\mu_k - 2iu + \frac{\pi i}{4}}] \\
&= \epsilon \sqrt{2} \prod_{k=1}^{L} t_k^{\frac{1}{2}} e^{\frac{\pi i}{4}(\mu_k - 1)} \frac{1}{2i} [e^{2i(u+\frac{i}{4}\log t_k - \frac{\pi}{8}\mu_k + \frac{\pi}{4})} - e^{-2i(u+\frac{i}{4}\log t_k - \frac{\pi}{8}\mu_k + \frac{\pi}{4})}] \\
&= \epsilon \sqrt{2}\, e^{\frac{\pi i}{4}[-L+\Sigma_{k=1}^{L}\mu_k]} \prod_{k=1}^{L} \sin 2\left(u + \frac{i}{4}\log t_k - \frac{\pi}{8}\mu_k + \frac{\pi}{4}\right) \\
&= R^{+} \prod_{k=1}^{P} \sin 2(u - u_k)
\end{aligned} \tag{3.66}$$

where $\epsilon^2 = \mu_k^2 = \prod_{k=1}^{L} \mu_k = 1$ and we have used (3.65). Suppose, for a given eigenvalue $T(u)$, that m is the number of μ_k such that $\mu_k = -1$. Since $\prod_{k=1}^{L} \mu_k = 1$, $m = 2\rho$ is even and

$$\sum_{k=1}^{L} \mu_k = -m + (L - m) = L - 2m = L - 4\rho \tag{3.67}$$

Hence

$$e^{\frac{\pi i}{4}(-L+\Sigma_{k=1}^{L}\mu_k)} = e^{-\pi i\rho} = (-1)^{\rho} \tag{3.68}$$

so that (3.66) becomes

$$T(u) = (-1)^{\rho}\, \epsilon \sqrt{2} [\sin 2(u - u_{\lceil \frac{L}{2} \rceil})]^{\delta_{L \bmod 2,\,1}} \prod_{k=1}^{\lfloor L/2 \rfloor} \sin 2(u - u_k^{+}) \sin 2(u - u_k^{-}) \tag{3.69}$$

where

$$u_k = \frac{\pi}{8}\,\mu_k - \frac{\pi}{4} - \frac{i}{4}\log t_k, \qquad k = 1, 2, \ldots, P \tag{3.70}$$

$$u_k^+ = u_k = \frac{\pi}{8}\,\mu_k - \frac{\pi}{4} - \frac{i}{4}\log t_k, \qquad k = 1, 2, \ldots, \lfloor P/2 \rfloor \tag{3.71}$$

$$u_k^- = u_{L+1-k} = \frac{\pi}{8}\,\bar{\mu}_k - \frac{\pi}{4} + \frac{i}{4}\log t_k, \qquad k = 1, 2, \ldots, \lfloor P/2 \rfloor \tag{3.72}$$

and

$$\bar{\mu}_k = \mu_{P+1-k}, \qquad k = 1, \ldots, \lfloor P/2 \rfloor \tag{3.73}$$

Here u_k^+, u_k^- are the zeros in the upper and lower half planes respectively. Note that, if $\mu_k = \bar{\mu}_k$, then u_k^- is simply the complex conjugate of u_k^+. Note also that real zeros $u_k \in \mathbb{R}$ occur for

$$k = \lceil P/2 \rceil, \qquad P \text{ odd} \tag{3.74}$$

Similarly, for $R = -1$, (2.26) becomes

$$\begin{aligned} T(u) &= \epsilon\sqrt{L}\,(2e^{2iu+\frac{\pi i}{4}})^{1-L} \prod_{k=1}^{L-1}\left(e^{4iu} + i\mu_k \tan\left(\frac{k\pi}{2L}\right)\right) \\ &= \epsilon\sqrt{L}\,e^{\frac{\pi i}{4}(1-L)} \prod_{k=1}^{L-1} \frac{1}{2}\,[e^{2iu} + t_k e^{-2iu+\frac{\pi i}{2}\mu_k}] \\ &= \epsilon\sqrt{L}\,e^{\frac{\pi i}{4}(1-L)} e^{\frac{\pi i}{4}\sum_{k=1}^{L-1}\mu_k} \prod_{k=1}^{L-1} \sin 2\left(u + \frac{i}{4}\log t_k - \frac{\pi}{8}\,\mu_k + \frac{\pi}{4}\right) \\ &= R^- \prod_{k=1}^{P} \sin 2(u - u_k) \end{aligned} \tag{3.75}$$

where $\epsilon^2 = \mu_k^2 = 1$ for all k and we have again used (3.65). But now

$$\exp\left[\frac{\pi i}{4}\left(1 - L + \sum_{k=1}^{L-1}\mu_k\right)\right] = \exp\left(-\frac{\pi i}{4}\,2m\right) = (-i)^m \tag{3.76}$$

where m is the number of μ_k such that $\mu_k = -1$. Hence (3.75) reduces to

$$T(u) = \epsilon(-i)^m \sqrt{L}\,[\sin 2(u - u_{\lceil \frac{L-1}{2} \rceil})]^{\delta_{L \bmod 2,\,0}} \prod_{k=1}^{\lfloor \frac{L-1}{2} \rfloor} \sin 2(u - u_k^+)\sin 2(u - u_k^-) \tag{3.77}$$

We want to pick out the eigenvalues of $\boldsymbol{T}(u)$ which are real for all $u \in \mathbb{R}$. First, suppose the zeros occur in complex conjugate pairs and $R^{\pm}$ is real. Then it is clear that $T(u)$ is real for all $u \in \mathbb{R}$. Conversely, suppose $T(u)$ is real for all $u \in \mathbb{R}$. Then

$$\overline{T(u)} = \overline{R^{\pm}} \prod_{k=1}^{P} \sin 2(u-\bar{u}_k) = T(u), \qquad u \in \mathbb{R} \tag{3.78}$$

which implies that the zeros u_k occur in complex conjugate pairs and that $R^{\pm}$ is real. Thus, $T(u)$ is real for all $u \in \mathbb{R}$ if and only if $R^{\pm}$ is real and the zeros u_k occur in complex conjugate pairs, that is, if the patterns of zeros in the upper and lower half planes are related by complex conjugation. Since $\varDelta$ and k are determined by the pattern of zeros in the upper half plane and $\bar{\varDelta}$ and $\bar{k}$ are determined by the pattern of zeros in the lower half plane, this picks out precisely the eigenvalues for which $\varDelta = \bar{\varDelta}$ and $k = \bar{k}$.[15] By (3.69) and (3.77), we see that the eigenvalue $T(u)$ is real for all $u \in \mathbb{R}$ if and only if

$$\mu_k = \bar{\mu}_k \qquad \text{and} \qquad \prod_{k=1}^{P} \mu_k = 1 \tag{3.79}$$

For $R = -1$ and L even (P odd), the last condition in (3.79) reduces to $\mu_{\lceil \frac{L-1}{2} \rceil} = 1$.

Using (3.79) and allowing for $\epsilon = \pm 1$, we can count the number r of real eigenvalues $T(u) \in \mathbb{R}$

$$r = 2(2^{\lfloor \frac{L}{2} \rfloor} + 2^{\lfloor \frac{L-1}{2} \rfloor}) = \begin{cases} 3(2^{L/2}), & L \text{ even} \\ 2^{(L+3)/2}, & L \text{ odd} \end{cases} \tag{3.80}$$

in agreement with (3.6). Since r is the net number of positive eigenvalues of the flip operator $\boldsymbol{F}$, as discussed in Section 3.1, any real eigenvector of $\boldsymbol{T}(u)$ is an eigenvector of $\boldsymbol{F}$ with $F = +1$. The remaining $2s$ eigenvectors occur in complex conjugate pairs with $F = \pm 1$. After cancellation of the contributions from the complex eigenvalues and removal of the trivial $\pm T(u)$ degeneracy, the partition function for the Ising model on the Klein bottle is

$$Z_{NM}^{\text{Klein}}(u) = \sum_n F_n D_n(u)^{M/2} = \sum_{n=0}^{r/2-1} D_n(u)^{M/2} = \sum_{n=0}^{r/2-1} |T_n(u)|^M \tag{3.81}$$

where the sum is over the $r/2$ real eigenvalues of $\boldsymbol{D}(u)$

$$D_n(u) = |T_n(u)|^2, \qquad n = 0, 1, \dots, r-1, \quad T_n(u) \in \mathbb{R} \tag{3.82}$$

3.6. Finitized Ising Partition Function on Klein Bottle

In this section, we follow ref. 16 to obtain the *finitized* conformal partition function $Z^{\text{Klein}}(L; q)$ in terms of finitized Virasoro characters.

We begin by using (2.27) to remove the degeneracy in the eigenvalue expressions (2.25) and (2.26) for the single row transfer matrix eigenvalues $T(u)$. This degeneracy is irrelevant for the eigenvalues $D(u) = \overline{T(u)}\, T(u)$ of the double row transfer matrix. Using (2.27) and (2.25) we find for $R = +1$

$$D(u) = \overline{T(u)}\, T(u) = 2^{-2L+1} \left| e^{4iu} + i\mu_{\lceil \frac{L}{2} \rceil} \tan\left(\frac{\pi(2\lceil L/2 \rceil - 1)}{4L}\right)\right|^{2\delta_{L \bmod 2,\, 1}}$$
$$\times \prod_{k=1}^{\lfloor L/2 \rfloor} \left| e^{4iu} + i\mu_k \tan\left(\frac{\pi(2k-1)}{4L}\right)\right|^2 \left| \bar{\mu}_k e^{4iu} + i\cot\left(\frac{\pi(2k-1)}{4L}\right)\right|^2 \tag{3.83}$$

with $\bar{\mu}_k = \mu_{L-k+1}$. Similarly, from (2.26) we find for $R = -1$

$$D(u) = \overline{T(u)}\, T(u) = 2^{2(1-L)} L \left| e^{4iu} + i\mu_{\lceil \frac{L-1}{2} \rceil} \tan\left(\frac{\pi \lceil \frac{L-1}{2} \rceil}{2L}\right)\right|^{2\delta_{L \bmod 2,\, 0}}$$
$$\times \prod_{k=1}^{\lfloor \frac{L-1}{2} \rfloor} \left| e^{4iu} + i\mu_k \tan\left(\frac{\pi k}{2L}\right)\right|^2 \left| \bar{\mu}_k e^{4iu} + i\cot\left(\frac{\pi k}{2L}\right)\right|^2 \tag{3.84}$$

with $\bar{\mu}_k = \mu_{L-k}$. We remark that although the forms for $T(u)$ are not unique due to the choice of ϵ, the forms given here for $D(u)$ are unique since they are independent of ϵ.

Taking the ratio of (3.83) and (3.84) with the largest eigenvalue $D_0(u)$ given by (3.83) with all $\mu_k = \bar{\mu}_k = 1$ and using the limits

$$\lim_{N \to \infty} N \log\left[\frac{e^{4iu} - i\tan(\frac{\pi K}{N})}{e^{4iu} + i\tan(\frac{\pi K}{N})}\right] = -2\pi i K \mathrm{e}^{-4iu}$$
$$\lim_{N \to \infty} N \log\left[\frac{-e^{4iu} + i\cot(\frac{\pi K}{N})}{e^{4iu} + i\cot(\frac{\pi K}{N})}\right] = 2\pi i K \mathrm{e}^{4iu} \tag{3.85}$$

along with the condition (3.79) and the definition of q in (3.56), we obtain

$$Z^{\text{Klein}}(L; q) = \sum_{\{\mu\}^+_{\lfloor L/2 \rfloor}} \prod_{k=1}^{\lfloor L/2 \rfloor} q^{2(k-1/2)\,\delta_{\mu_k, -1}} + \sum_{\{\mu\}^-_{\lfloor L/2 \rfloor}} \prod_{k=1}^{\lfloor L/2 \rfloor} q^{2(k-1/2)\,\delta_{\mu_k, -1}}$$
$$+ |q|^{1/8} \sum_{\{\mu\}_{\lfloor (L-1)/2 \rfloor}} \prod_{k=1}^{\lfloor (L-1)/2 \rfloor} q^{2k\delta_{\mu_k, -1}} \tag{3.86}$$

Here the factor $|q|^{1/8}$ arises from the scaling limit of the ratio of the largest eigenvalue in (3.84), which has all $\mu_k = \bar{\mu}_k = 1$, to the overall largest eigenvalue $D_0(u)$ in (3.83) which has all $\mu_k = \bar{\mu}_k = 1$. This ratio can be computed using the Euler–Maclaurin formula as in Appendix B. Using the expressions for the finitized Virasoro characters, we finally obtain the *finitized* partition function for the Ising model on the Klein bottle

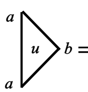
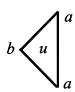
$$Z^{\text{Klein}}(L;q) = X_0\left(\left\lfloor \frac{L}{2} \right\rfloor; q^2\right) + |q|\, X_{1/2}\left(\left\lfloor \frac{L}{2} \right\rfloor; q^2\right) + |q|^{1/8}\, X_{1/16}\left(\left\lfloor \frac{L-1}{2} \right\rfloor; q^2\right) \tag{3.87}$$

Notice that if we set $q = 1$, then $Z(L;1) = r/2$ as given by (3.80) corresponding to the counting of the number of real eigenvalues left after cancellation of the complex conjugate pairs. As for the torus, the even sector of the spectra with $R = 1$ is given by X_0 and $X_{1/2}$ and the odd sector with $R = -1$ is given by $X_{1/16}$.

4. ISING MODEL ON THE MÖBIUS STRIP

In this section we consider the Ising model on the Möbius strip. Consider a square lattice consisting of a strip of N columns and M rows. Along the vertical direction we apply a flip operator $\boldsymbol{F}$ before joining row M to row 1. We apply integrable boundary conditions[21] on the left and right edges of the strip. These boundary conditions labelled by $+$, $-$, F (free) are specified by boundary triangle Boltzmann weights on the left and right edges of the double row transfer matrix $\boldsymbol{D}(u)$. We build up the Möbius strip by M applications of the double row transfer matrix. For compatibility with the flip the boundary conditions on the left and right must be the same. This is because the Möbius strip only has one edge. By the spin reversal symmetry, the partition function with $+$ boundary conditions is the same as the partition function with $-$ boundary conditions so we do not need to consider $-$ boundary conditions.

The left and right boundary weights are

$$B\left(\begin{matrix} a & \\ & b \\ a & \end{matrix}\middle|\, u\right) =$$  $$= B\left(\begin{matrix} & a \\ b & \\ & a \end{matrix}\middle|\, u\right) =$$  (4.1)

where it is understood that these weights vanish if the adjacency condition is not satisfied along the edges (a, b). These weights must satisfy the boundary Yang–Baxter equation. For the Ising model, we study the boundary conditions specified by the following non-zero weights

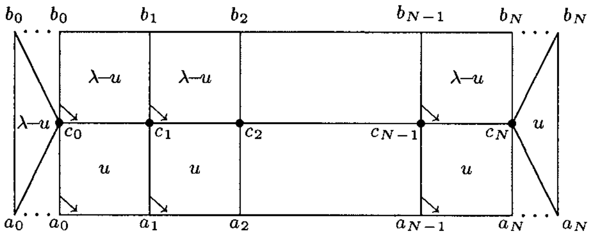
- *Fixed* +: The edge spins are fixed to $a=1$

$$B\begin{pmatrix}1 & \\ & 2 \\ 1 & \end{pmatrix}\!\!\Bigg|\,u\Bigg) = B\Bigg(2 \begin{matrix}1\\ \\ 1\end{matrix}\Bigg|\,u\Bigg) = 1 \tag{4.2}$$

- *Free F*: The edge spins are fixed to $a=2$

$$B\Bigg(\begin{matrix}2\\ \\ 2\end{matrix}\; c\,\Bigg|\,u\Bigg) = B\Bigg(c\; \begin{matrix}2\\ \\ 2\end{matrix}\Bigg|\,u\Bigg) = \frac{1}{\sqrt{2}}, \qquad c = 1, 3 \tag{4.3}$$

For a strip with N columns, the double row transfer matrix $\boldsymbol{D}(u)$ with rows $\boldsymbol{a} = \{a_0,..., a_N\}$ and $\boldsymbol{b} = \{b_0,..., b_N\}$ is defined diagrammatically[21] by

$\boldsymbol{D}(u)_{\boldsymbol{a},\boldsymbol{b}} =$  (4.4)

For the Ising model on the Möbius strip we require $a_0 = b_0 = a_N = b_N$. The double row transfer matrices are real symmetric and crossing symmetric[21]

$$\boldsymbol{D}(u) = \boldsymbol{D}^T(u) = \boldsymbol{D}(\lambda - u), \qquad u \in \mathbb{R} \tag{4.5}$$

The dimension of the transfer matrices $\boldsymbol{D}(u)$ are given in terms of the adjacency matrix A by

$$\dim \boldsymbol{D}(u) = [A^N]_{a_0, a_0} = \begin{cases} 2^{L-1}, & \text{fixed} + \\ 2^L, & \text{free } F \end{cases} \tag{4.6}$$

where $N = 2L$ must be even for both boundary conditions.

The flip operator $\boldsymbol{F}$ satisfying $\dim \boldsymbol{F} = \dim \boldsymbol{D}(u)$, $\boldsymbol{F}^2 = \boldsymbol{I}$, $\boldsymbol{F}^T = \boldsymbol{F}$ is defined by

$$\boldsymbol{F}_{\boldsymbol{a},\boldsymbol{b}} = \prod_{j=0}^{N} \delta_{a_j, b_{N-j}} \tag{4.7}$$

Proceeding as for the Klein Bottle, the number of negative eigenvalues of $\boldsymbol{F}$ for the Möbius strip is

$$2s_a = [A^N]_{a,a} - \sum_{k=1}^{3} [A^{N/2}]_{a,k} \tag{4.8}$$

where the edge spin is $a=1$ for fixed $+$ boundaries and $a=2$ for free F boundaries.

The matrices $\boldsymbol{D}(u)$ commute with $\boldsymbol{F}$. In fact, using (2.7) and (4.5) we find

$$\boldsymbol{F}\boldsymbol{D}(u)\,\boldsymbol{F} = \boldsymbol{D}^T(\lambda-u) = \boldsymbol{D}^T(u) = \boldsymbol{D}(u) \tag{4.9}$$

The partition function is thus

$$Z_{MN}^{\text{Möbius}}(u) = \text{Tr}[\boldsymbol{F}\boldsymbol{D}(u)^{M/2}] = \sum_n F_n D_n(u)^{M/2} \tag{4.10}$$

where the sum is over all eigenvalues.

More generally, for other lattice models with integrable boundary conditions that are non-diagonal, the condition $a_0 = b_0 = a_N = b_N$ is not satisfied and the last equality in (4.9) does not hold. In such cases it can be shown[3] that $\boldsymbol{D}(u)$ is similar to a symmetric matrix $\widetilde{\boldsymbol{D}}(u)$ that does commute with $\boldsymbol{F}$. Let us define the diagonal matrices

$$\mathscr{A}_{a,b} = \left(\frac{S_{a_N}}{S_{a_0}}\right)^{1/4} \prod_{j=0}^{N} \delta_{a_j,b_j}, \qquad \mathscr{A}^{-1}_{a,b} = \left(\frac{S_{a_0}}{S_{a_N}}\right)^{1/4} \prod_{j=0}^{N} \delta_{a_j,b_j} \tag{4.11}$$

so that

$$\boldsymbol{F}\mathscr{A}\boldsymbol{F} = \mathscr{A}^{-1}, \qquad \mathscr{A}\boldsymbol{F} = \boldsymbol{F}\mathscr{A}^{-1}, \qquad \boldsymbol{F}\mathscr{A} = \mathscr{A}^{-1}\boldsymbol{F} \tag{4.12}$$

Let us also define

$$\widetilde{\boldsymbol{D}}(u) = \mathscr{A}\boldsymbol{D}(u)\,\mathscr{A}^{-1} \tag{4.13}$$

Then, by using (2.7) we find

$$\widetilde{\boldsymbol{D}}^T(u) = \widetilde{\boldsymbol{D}}(u) \tag{4.14}$$

so that $\widetilde{D}(u)$ is real symmetric for $u \in \mathbb{R}$. It then also follows using the valid part of (4.9), (4.12) and (4.14) that F and $\widetilde{D}(u)$ commute

$$\begin{aligned} F\widetilde{D}(u)\, F &= F\mathscr{A} D(u)\, \mathscr{A}^{-1} F = \mathscr{A}^{-1} F D(u)\, F \mathscr{A} \\ &= \mathscr{A}^{-1} D^T(u)\, \mathscr{A} = (\mathscr{A} D(u)\, \mathscr{A}^{-1})^T = \widetilde{D}^T(u) = \widetilde{D}(u) \end{aligned} \tag{4.15}$$

In this case $D(u)$ should be replaced by $\widetilde{D}(u)$ in the partition function (4.10).

4.1. Finitized Ising Partition Function on the Möbius Strip

The transfer matrices $D(u)$ for both fixed and free boundary conditions form[21] a commuting family $D(u)\, D(v) = D(v)\, D(u)$ whose eigenvalues $D(u)$ satisfy[16] the functional equation

$$D(u)\, D(u+\lambda) = \frac{\cos^{2(N+1)} 2u - \sin^{2(N+1)} 2u}{\cos 4u} \tag{4.16}$$

where $\lambda = \pi/4$ is the crossing parameter. For free boundary conditions the solution of this functional equation yields

$$D(u) = 2^{-L} \prod_{k=1}^{L} \left(\operatorname{cosec}\left(\frac{\pi(2k-1)}{2(2L+1)} \right) + \mu_k \sin 4u \right) \tag{4.17}$$

where $N = 2L$ and $\mu_k = \pm 1$ so that there are 2^L eigenvalues. The same solution applies for fixed + boundary conditions but with the constraint

$$\prod_{k=1}^{L} \mu_k = 1 \tag{4.18}$$

In this case there are 2^{L-1} eigenvalues in agreement with (4.6). The eigenvalues $D(u)$ are thus specified by the set $\{\mu_k\}$.

The asymptotic behaviour of the finite-size partition function Z_{NM} of the critical Ising model on the Möbius strip in the limit of large N and M with the aspect ratio M/N fixed is given by

$$Z_{NM}^{\text{Möbius}}(u) = \operatorname{Tr}[F D(u)^{M/2}] \sim \exp[-NM f_{\text{bulk}}(u) - M f_{\text{bdy}}(u)]\, Z^{\text{Möbius}}(q) \tag{4.19}$$

where $D(u)$ is the double row transfer matrix, $f_{\text{bulk}}(u)$ is the bulk free energy, $f_{\text{bdy}}(u)$ is the boundary free energy and $Z^{\text{Möbius}}(q)$ is the universal conformal partition function. The leading corrections to each transfer matrix eigenvalue $D(u)$ are of the form

$$\frac{1}{2}\log D(u) = -N f_{\text{bulk}}(u) - f_{\text{bdy}}(u) + \frac{\pi}{N}\left(\frac{c}{24} - \Delta - k\right)\sin 4u + o\left(\frac{1}{N}\right) \tag{4.20}$$

with $\Delta = 0, 1/2, 1/16$ and k a non-negative integer. On the Möbius strip we define

$$q = \exp\left(-\pi \frac{M}{N}\sin 4u\right) = |q_{\text{Torus}}|^{1/2} \tag{4.21}$$

where q_{Torus} is the modular paramter on the torus (2.15). The *finitized* conformal partition function is thus

$$Z^{\text{Möbius}}(L; q) = \sum_n F_n \left(\frac{D_n}{D_0}\right)^{M/2} \tag{4.22}$$

where $n = 0, 1,...$ labels the eigenvalues D_n of $\boldsymbol{D}(u)$ and $F_n = \pm 1$ of $\boldsymbol{F}$ for N columns and D_0 is the largest eigenvalue with all $\mu_k = 1$.

Let m be the number of μ_k such that $\mu_k = -1$. Assuming that m remains finite as $N \to \infty$, we see that

$$\lim_{N\to\infty} N \log\left(\frac{D(u)}{D_0(u)}\right) = -\pi \sin 4u \sum_{k=1}^{L} (2k-1)\, \delta_{\mu_k, -1} \tag{4.23}$$

Hence

$$\left(\frac{D(u)}{D_0(u)}\right)^{M/2} \simeq \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k,-1}} \tag{4.24}$$

where q is given by (4.21). So the finitized partition function is

$$Z^{\text{Möbius}}(L; q) = \begin{cases} \sum_{\{\mu\}_L^{+}} F \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k,-1}} & \text{Fixed } + \\ \sum_{\{\mu\}_L} F \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k,-1}} & \text{Free } F \end{cases} \tag{4.25}$$

where $\{\mu\}_L^{\pm}$ denotes the sequences restricted by $\prod_{k=1}^{L} \mu_k = \pm 1$.

From ref. 16, the eigenvalues (4.17) can be written as

$$D(u) = \epsilon(4ie^{4iu})^{-L} \prod_{\substack{k=1\\k\neq L+1}}^{2L+1} \left(e^{4iu} + i\mu_k \tan\left(\frac{\pi(2k-1)}{4(2L+1)}\right)\right) \tag{4.26}$$

with $\epsilon = \prod_{k=1}^{L} \mu_k$. As in Section 3.5, these eigenvalues are determined by their zeros in the complex u-plane as given by (3.62). Since the zeros occur in complex conjugate pairs we will only look at the zeros of $D(u)$ in the upper-half plane given by

$$u_k = \frac{\pi}{8}\bar{\mu}_k - \frac{\pi}{4} - \frac{i}{4}\log t_{L+1-k} \mod \frac{\pi}{2}, \qquad k = 1, 2,..., L \tag{4.27}$$

where $\bar{\mu}_k = \mu_{L+1-k}$ and $t_k = \tan\left(\frac{\pi(2k-1)}{4(2L+1)}\right)$. The zeros occur on certain fixed lines and are classified into 1-strings and 2-strings. The relevant analyticity strip is $-\pi/8 \leqslant \mathrm{Re}(u) \leqslant 3\pi/8$ and a 1-string is a single zero u_k such that

$$\mathrm{Re}(u_k) = \frac{\pi}{8} \tag{4.28}$$

A 2-string is a pair of zeros (u_k, u_k') with equal imaginary part and

$$(\mathrm{Re}(u_k), \mathrm{Re}(u_k')) = \left(-\frac{\pi}{8}, \frac{3\pi}{8}\right) \tag{4.29}$$

A distribution of zeros for a typical eigenvalue $D(u)$ for $L = 8$ is depicted in Fig. 3.

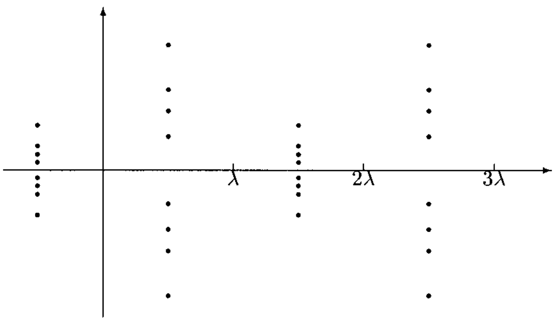


Fig. 3. Zeros of a typical eigenvalue of $D(u)$ in the period strip $-\lambda < \mathrm{Re}(u) \leqslant 3\lambda$ in the complex u-plane. The relevant analyticity strip is $-\lambda/2 \leqslant \mathrm{Re}(u) \leqslant 3\lambda/2$.

The eigenvalues are completely classified by the string contents and the relative orderings of the strings in the analyticity strip. For each eigenvalue $D(u)$, there are $L = N/2$ distinct strings on the upper-half complex u-plane. Let m be the number of 1-strings in the analyticity strip and n the number of 2-strings. Then the string contents m and n satisfy the simple (m, n) system[22, 20]

$$m + n = L \tag{4.30}$$

Note that $j = 1$ labels the 1- and 2-strings closest to the real axis and for each eigenvalue the ordering of the imaginary parts of the strings is

$$\mathrm{Im}(u_L) > \mathrm{Im}(u_{L-1}) > \cdots > \mathrm{Im}(u_2) > \mathrm{Im}(u_1) > 0 \tag{4.31}$$

From (4.27), a string at $\mathrm{Im}(u_k)$ is a 1-string if $\bar{\mu}_k = -1$ and a 2-string if $\bar{\mu}_k = 1$. Suppose that

$$\bar{\mu}_k = -1, \qquad k = k_1, k_2, \ldots, k_m, \qquad k_1 < k_2 < \ldots < k_m \tag{4.32}$$

Then we define the quantum numbers

$$I_j = \sum_{k = k_j}^{L} \tfrac{1}{2}(1 + \bar{\mu}_k) = \{\#\text{ of 2-strings above given 1-string } k_j\} \tag{4.33}$$

Alternatively, these can be defined recursively by

$$I_j - I_{j-1} + k_j - k_{j-1} = 1, \qquad j = 1, 2, \ldots, m \tag{4.34}$$

with $k_0 = 0$, $I_0 = L - m$ and solution

$$I_j = L - m + j - k_j, \qquad j = 0, 1, \ldots, m \tag{4.35}$$

The quantum numbers $I = (I_1, I_2, \ldots, I_m)$ satisfy

$$L - m = n \geqslant I_1 \geqslant I_2 \geqslant \cdots \geqslant I_m \geqslant 0 \tag{4.36}$$

and uniquely label all of the eigenvalues with given string content m.

The commuting matrices $\boldsymbol{D}(u)$ and $\boldsymbol{F}$ have a common set of eigenvectors. If the eigenvector $\boldsymbol{x}$ associated with the eigenvalue $D(u)$ satisfies $\boldsymbol{F}\boldsymbol{x} = \boldsymbol{x}$ we say that the eigenvalue $D(u)$ has parity $F = +1$ under the flip. Otherwise, if the eigenvector $\boldsymbol{x}$ satisfies $\boldsymbol{F}\boldsymbol{x} = -\boldsymbol{x}$, we say that the eigenvalue $D(u)$ has parity $F = -1$ under the flip. For both fixed and free boundary conditions, we find the following results from numerics:

1. An eigenvalue $D(u)$ with quantum numbers $(I_1, I_2, \ldots, I_m) = (0, 0, \ldots, 0)$, such that the 1-strings are all further from the real axis than the 2-strings, has parity $F = +1$ under the flip $\boldsymbol{F}$.

2. An eigenvalue $D(u)$ with quantum numbers $(I_1, I_2, \ldots, I_m)$ has parity

$$F = (-1)^{I_1 + I_2 + \cdots + I_m} \tag{4.37}$$

under the flip $\boldsymbol{F}$. This implies that interchanging the position of a 1-string with the position of an adjacent 2-string changes the sign of the parity.

These parity properties can not be obtained by studying the eigenvalues alone. Rather these parity properties reflect deep properties of the eigenvectors which we do not study here. Nevertheless, these observations allow the flip eigenvalue $F = \pm 1$ to be read off directly from the eigenvalue $D(u)$ just by looking at the locations of its zeros in the complex u-plane.

For fixed + boundaries on the Möbius strip, it now follows from (4.25) and our numerical observations that

$$\begin{aligned} Z^+(L; q) &= \sum_{\{\mu\}_L^+} F \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k, -1}} = \sum_{m \text{ even}} q^{\frac{m^2}{2}} \sum_{I} (-q)^{\sum_{j=1}^m I_j} \\ &= \sum_{m \text{ even}} q^{\frac{m^2}{2}} \begin{bmatrix} L \\ m \end{bmatrix}_{(-q)} = X_0(L; -q) \end{aligned} \tag{4.38}$$

where we have used (2.32) and the facts that m is even, $\mu_k = \bar{\mu}_{L+1-k}$ and

$$\sum_{k=1}^{L} \left(k - \frac{1}{2}\right) \delta_{\mu_k, -1} = \sum_{j=1}^{m} \left(L + 1 - k_j - \frac{1}{2}\right) = \frac{m^2}{2} + \sum_{j=1}^{m} I_j \tag{4.39}$$

Similarly, for free boundaries on the Möbius strip, the finitized partition function is

$$\begin{aligned} Z^F(L; q) &= \sum_{\{\mu\}_L^+} F \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k, -1}} + \sum_{\{\mu\}_L^-} F \prod_{k=1}^{L} q^{(k-1/2)\,\delta_{\mu_k, -1}} \\ &= \sum_{m \text{ even}} q^{\frac{m^2}{2}} \sum_{I} (-q)^{\sum_{j=1}^m I_j} + q^{1/2} \sum_{m \text{ odd}} q^{\frac{m^2-1}{2}} \sum_{I} (-q)^{\sum_{j=1}^m I_j} \\ &= X_0(L; -q) + q^{1/2} X_{1/2}(L; -q) \end{aligned} \tag{4.40}$$

If we let $L \to \infty$ we obtain the conformal partition functions of the Ising model on the Möbius strip in terms of Virasoro characters. Notice also that removing the flip F in (4.38) and (4.40) gives the finitized cylinder partition functions $Z_{NM}^{+|+}(q)$ and $Z_{NM}^{F|F}(q)$ respectively. For the Möbius strip with free boundary conditions, the even sector of the spectra with $R=1$ is given by X_0 and the odd sector with $R=-1$ is given by $X_{1/2}$.

Lastly we note that, in agreement with (4.8), the number of negative eigenvalues of $\boldsymbol{F}$ is

$$\lim_{q\to 1}\tfrac{1}{2}[Z^{a|a}(L;q)-Z^{a}(L;q)]=\begin{cases}\tfrac{1}{2}(2^{L-1}-2^{\lfloor\frac{L}{2}\rfloor}), & a=1 \text{ or } +\\ \tfrac{1}{2}(2^{L}-2^{\lfloor\frac{L+1}{2}\rfloor}), & a=2 \text{ or } F\end{cases}\tag{4.41}$$

where we have used

$$\frac{1}{2}\left(\begin{bmatrix}L\\ m\end{bmatrix}_{q=1}-\begin{bmatrix}L\\ m\end{bmatrix}_{q=-1}\right)=\begin{cases}\dfrac{1}{2}\dbinom{L}{m}, & L \text{ odd, } m \text{ even}\\ \dfrac{1}{2}\left[\dbinom{L}{m}-\dbinom{\lfloor\frac{L}{2}\rfloor}{\lfloor\frac{m}{2}\rfloor}\right], & \text{otherwise}\end{cases}\tag{4.42}$$

5. SUMMARY AND DISCUSSION

In this paper we have obtained the finitized conformal partitions of the Ising model on the Klein bottle and Möbius strip using Yang–Baxter techniques and the solution of functional equations. Changing the topology from the torus to the Klein bottle or the cylinder to the Möbius strip is achieved by inserting a flip operator before gluing the boundaries with the trace. Under the action of the flip operator, a parity $F=\pm 1$ is assigned to the eigenvalues of the commuting transfer matrices. The task is thus reduced to finding the parities of the eigenvalues of the relevant transfer matrices. Although in this paper we have only considered the Ising model, our methods and observations apply more generally to solvable lattice models as we now explain.

For the Klein bottle, it was shown analytically that all the complex eigenvalues of the single row transfer matrix cancel with their complex conjugates. The partition function for Ising model on the Klein bottle is thus given by a projection from the partition function on the torus

$$\begin{aligned}Z^{\text{Torus}}(q) &= |\chi_0(q)|^2+|\chi_{1/2}(q)|^2+|\chi_{1/16}(q)|^2\\ &\mapsto \chi_0(q^2)+\chi_{1/2}(q^2)+\chi_{1/16}(q^2) = Z^{\text{Klein}}(q)\end{aligned}\tag{5.1}$$

where only the left-right chiral symmetric part is retained. In fact, from numerics, this projection is valid for any A_L model with $L \geqslant 3$ so that we observe the projection

$$Z_{A_L}^{\text{Torus}}(q) = \sum_{\Delta} |\chi_\Delta(q)|^2 \mapsto \sum_{\Delta} \chi_\Delta(|q|^2) = Z_{A_L}^{\text{Klein}}(q) \tag{5.2}$$

where the sum is over the Kac table. Similarly, we have checked that the same mechanism works for the D_4 or Potts model

$$\begin{aligned} Z_{D_4}^{\text{Torus}}(q) &= |\chi_0(q)+\chi_3(q)|^2+|\chi_{2/5}(q)+\chi_{7/5}(q)|^2+2\,|\chi_{1/15}(q)|^2+2\,|\chi_{2/3}(q)|^2 \\ &\mapsto \chi_0(q^2)+\chi_3(q^2)+\chi_{2/5}(q^2)+\chi_{7/5}(q^2)+2\chi_{1/15}(q^2)+2\chi_{2/3}(q^2) \\ &= Z_{D_4}^{\text{Klein}}(q) \end{aligned} \tag{5.3}$$

For the Möbius strip, the modular parameter effectively changes from q to $-q$ under the action of the flip operator which transforms the topology from the cylinder to the Möbius strip. More specifically, since $q = \exp(2\pi i\tau)$, this is equivalent to saying that $\tau \mapsto \tau + 1/2$. Again we find numerically that this projection mechanism applies to other A_L models such as, for example, A_4 with the $(r, s) = (2, 2)$ boundary condition

$$\begin{aligned} Z_{A_4}^{\text{Cyl},\,(2,2)}(q) &= \chi_{1,1}(q)+\chi_{1,3}(q)+\chi_{3,1}(q)+\chi_{3,3}(q) \\ &\mapsto \tilde{\chi}_{1,1}(-q)+\tilde{\chi}_{1,3}(-q)+\tilde{\chi}_{3,1}(-q)+\tilde{\chi}_{3,3}(-q) \\ &= Z_{A_4}^{\text{Möbius},\,(2,2)}(q) \end{aligned} \tag{5.4}$$

where the tilde on the Virasoro characters means that the leading fractional power of q is replaced by the same fractional power of $|q|$.

Our results for the Ising model are in agreement with refs. 7 and 9. To compare the results with ref. 7, we use the identities

$$\sqrt{\frac{\theta_3}{\eta}} = q^{-1/48} \prod_{n=0}^{\infty} (1+q^{n+1/2}), \qquad \sqrt{\frac{\theta_4}{\eta}} = q^{-1/48} \prod_{n=0}^{\infty} (1-q^{n+1/2}) \tag{5.5}$$

and $\tau = i\,\frac{R}{4L} + 1/2$ to re-express $Z_{\text{fixed}(\pm)}(\tau)$ and $Z_{\text{free}}^{(1)}(\tau)$ in (11) and (12) of ref. 7 from theta and eta functions to q-product representation (note that $Z_{\text{free}}^{(2)}(\tau)$ corresponds to anti-periodic boundary which we don't consider here).

In ref. 9, again we first re-express the theta functions into q-product representaion. For Klein bottle, we use the fact that $\prod_{n=1}^{N} (1-q^{2n-1})(1+q^n) = 1 + O(q^{N+1})$, and substituting the aspect ratio ξ into q, we obtain the same exact formula (5.1) in terms of q-products (2.20,2.21,2.21). For Möbius strip, we confirm the result by series expansion about $q=0$ up to 100 terms using *Mathematica*.

Finally, we remark that it would be of interest to generalize our results on the Klein bottle and Möbius strip to the more general twisted boundary conditions discussed recently in the context of conformal field theory by Petkova and Zuber.[2]

APPENDIX A: SOME LINEAR ALGEBRA

We list some useful properties of $\boldsymbol{T}(u)$ and $\boldsymbol{F}$:

1. If $\boldsymbol{T}(u)\,\boldsymbol{x} = T(u)\,\boldsymbol{x}$ and $\boldsymbol{T}(u)$ is real then

$$\boldsymbol{T}(u)\,\bar{\boldsymbol{x}} = \overline{\boldsymbol{T}(u)\,\boldsymbol{x}} = \overline{T(u)}\,\bar{\boldsymbol{x}} \tag{A.1}$$

so that if $\boldsymbol{x}$ is real then $\overline{T(u)} = T(u)$ and $T(u)$ is real.

2. If $\boldsymbol{T}(u)$ is a real normal matrix with $\boldsymbol{T}^T(u) = \boldsymbol{T}(\lambda-u)$ then the eigenvalue of $\boldsymbol{T}^T(u) = \boldsymbol{T}(\lambda-u)$ is $\overline{T(u)}$. This follows since, if $\boldsymbol{x}$ is a common eigenvector of $\boldsymbol{T}(u)$ and $\boldsymbol{T}(\lambda-u)$ so that $\boldsymbol{T}(u)\,\boldsymbol{x} = T(u)\,\boldsymbol{x}$ and $\boldsymbol{T}(\lambda-u)\,\boldsymbol{x} = T(\lambda-u)\,\boldsymbol{x}$, then

$$T(\lambda-u)\,|\boldsymbol{x}|^2 = \boldsymbol{x}^\dagger \boldsymbol{T}(\lambda-u)\,\boldsymbol{x} = \boldsymbol{x}^\dagger \boldsymbol{T}^T(u)\,\boldsymbol{x} = (\boldsymbol{T}(u)\,\bar{\boldsymbol{x}})^T\,\boldsymbol{x} = \overline{T(u)}\,|\boldsymbol{x}|^2 \tag{A.2}$$

and so $T(\lambda-u) = \overline{T(u)}$.

3. Suppose further that T_i and T_j are eigenvalues of $\boldsymbol{T} = \boldsymbol{T}(u_0)$ and $T_i \neq T_j$, then the corresponding eigenvectors $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are orthogonal. This follows since

$$\boldsymbol{x}_i^\dagger(\boldsymbol{T}\boldsymbol{x}_j) = T_j \boldsymbol{x}_i^\dagger \boldsymbol{x}_j = (\boldsymbol{T}^T \boldsymbol{x}_i)^\dagger\,\boldsymbol{x}_j = (\overline{T_i}\,\boldsymbol{x}_i)^\dagger \boldsymbol{x}_j = T_i\,\boldsymbol{x}_i^\dagger \boldsymbol{x}_j \tag{A.3}$$

which implies $(T_i - T_j)\,\boldsymbol{x}_i^\dagger \boldsymbol{x}_j = 0$ and $\boldsymbol{x}_i^\dagger \boldsymbol{x}_j = 0$.

4. For any eigenvector $\boldsymbol{x}$ of $\boldsymbol{T}(u)$, $\boldsymbol{F}\boldsymbol{x}$ is also an eigenvector of $\boldsymbol{T}(u)$ with the corresponding eigenvalue $\overline{T(u)}$:

$$\boldsymbol{T}(u)(\boldsymbol{F}\boldsymbol{x}) = (\boldsymbol{T}(u)\,\boldsymbol{F})\,\boldsymbol{x} = (\boldsymbol{F}\boldsymbol{T}(\lambda-u))\boldsymbol{x} = \overline{T(u)}\,\boldsymbol{F}\boldsymbol{x} \tag{A.4}$$

5. By (3.16) and the normality of $\boldsymbol{T}(u)$, $\boldsymbol{F}$ and $\boldsymbol{D}(u) = \boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)$ commute:

$$\boldsymbol{F}(\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u)) = \boldsymbol{T}(u)\,\boldsymbol{F}\boldsymbol{T}(u) = \boldsymbol{T}(u)\,\boldsymbol{T}(\lambda-u)\,\boldsymbol{F} = (\boldsymbol{T}(\lambda-u)\,\boldsymbol{T}(u))\,\boldsymbol{F} \tag{A.5}$$

APPENDIX B: EULER–MACLAURIN

For the single row transfer matrix $\boldsymbol{T}(u)$, we compute the scaling limit of the largest eigenvalues in the $R=+1$ and $R=-1$ sectors by applying the midpoint and endpoint Euler–Maclaurin formulas

$$\sum_{k=1}^{m} f(a+(2k-1)\,h)=\frac{1}{h}\int_a^b f(t)\,dt-\frac{h}{24}\,[f'(b)-f'(a)]+\mathrm{O}(h^2) \tag{B.1}$$

$$\sum_{k=0}^{m} f(a+kh)=\frac{1}{h}\int_a^b f(t)\,dt+\frac{1}{2}\,[f(b)+f(a)] \\ +\frac{h}{12}\,[f'(b)-f'(a)]+\mathrm{O}(h^2) \tag{B.2}$$

where $b=a+mh$. These formulas are valid if $f(t)$ is twice differentiable on the closed interval $[a, b]$.

In the $R=+1$ sector, the logarithm of the largest eigenvalue can be written as

$$\log T_0(u)=\frac{1}{2}\,(1-L)\log 2+\frac{1}{2}\sum_{k=1}^{L}\log\left[\sin 4u+\csc\left(\frac{\pi(2k-1)}{2L}\right)\right] \tag{B.3}$$

Suppressing the dependence on u, we define the function

$$f(t)=\log[t(\pi-t)(\sin 4u+\csc t)] \tag{B.4}$$

so that

$$\log T_0(u)=\frac{1}{2}\,(1-L)\log 2+\sum_{k=1}^{L} f\left(\frac{\pi(2k-1)}{2L}\right)-2\sum_{k=1}^{L}\log\left(\frac{\pi(2k-1)}{2L}\right) \tag{B.5}$$

We define f in this way to remove the singularities of $\csc t$ at $t=0, \pi$. Applying the midpoint Euler–Maclaurin formula and the asymptotic expansion of the gamma function

$$\log \Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log 2\pi+\frac{1}{12z}+\mathrm{O}(z^{-2}) \tag{B.6}$$

we obtain

$$\log T_0(u)=\frac{N}{4\pi}\int_0^{\pi}\log\frac{1}{2}\,(\sin 4u+\csc t)\,dt+\frac{\pi}{12N}\sin 4u+\mathrm{o}\left(\frac{1}{N}\right) \tag{B.7}$$

which agrees with (2.16) with $c=1/2$, $\Delta=\bar{\Delta}=k=\bar{k}=0$ and bulk free energy

$$f_{\rm bulk}=-\frac{1}{4\pi}\int_0^\pi \log\frac{1}{2}\,(\sin 4u+\csc t)\,dt \tag{B.8}$$

Similarly, using the endpoint Euler–Maclaurin formula in the $R=-1$ sector, the logarithm of the largest eigenvalue is

$$\begin{aligned}\log T_1(u)&=\frac{1}{2}\log L+\frac{1-L}{2}\log 2+\frac{1}{2}\sum_{k=1}^{L-1}\log\left[\sin 4u+\csc\left(\frac{\pi k}{L}\right)\right]\\&=\frac{N}{4\pi}\int_0^\pi \log\frac{1}{2}\,(\sin 4u+\csc t)\,dt-\frac{\pi}{6N}\sin 4u+{\rm o}\left(\frac{1}{N}\right)\end{aligned} \tag{B.9}$$

with $f_{\rm bulk}$ defined by (B.8) and $\Delta=1/16$. Hence we conclude that

$$\log[T_1(u)/T_0(u)]=-\frac{\pi}{4N}\sin 4u+{\rm o}\left(\frac{1}{N}\right) \tag{B.10}$$

and we obtain the factor $|q|^{1/8}$ in (3.87).

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